Directed Sets of Normal Bases

This section develops some properties of directed sets of normal bases, especially in the case of a discrete space. The main result: For an infinite discrete space, any compactification, which is a supremum of finite-point compactifications or, equivalently, which is zero-dimensional, can be obtained from a normal basis via the Wallman-Frink construction. Notation and definitions from [6] will be used without additional reference here.

Nested Normal Bases

Lemma R9.1.1 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Then

i) If $\mathcal{F}$ is a $\mathcal{Z}_2$-filter, then $\mathcal{F} \cap \mathcal{Z}_1$ is a $\mathcal{Z}_1$-filter.

ii) For every $x \in X$, ${\mathcal{F}_x}^2 \cap \mathcal{Z}_1 = {\mathcal{F}_x}^1$.

iii) Assume for every $\mathcal{F}$ in $\omega(\mathcal{Z}_2)$ that $\mathcal{F} \cap \mathcal{Z}_1$ is in $\omega(\mathcal{Z}_1)$.

Then $[(\omega(\mathcal{Z}_2), \iota_2)] \geq [(\omega(\mathcal{Z}_1), \iota_1)]$.

Proof: Conclusions i) and ii) are straightforward consequences of the definitions. The added assumption of iii) allows definition of $h : \omega(\mathcal{Z}_2) \to \omega(\mathcal{Z}_1)$ by $h(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_1$. Let $\mathcal{S}$ be closed in $\omega(\mathcal{Z}_1)$ and let $\mathcal{F} \notin h^{-1}[\mathcal{S}]$. Then $h(\mathcal{F}) \notin \mathcal{S}$ and so there is $Z \in \mathcal{Z}_1$ such that $\mathcal{S} \subseteq Z^\omega$ and $Z \notin h(\mathcal{F})$. By the definition of h, $Z \notin \mathcal{F}$ and $h^{-1}[\mathcal{S}] \subseteq Z^\omega$ and thus $h^{-1}[\mathcal{S}]$ is closed. Clearly from ii) $h \circ \iota_2 = \iota_1$, and so the dense $\iota_1[X]$ is contained in $h[\omega(\mathcal{Z}_2)]$, which is closed by the continuity of h and the compactness of $\omega(\mathcal{Z}_2)$. Thus h is the continuous surjection required to show the claimed inequality.

Definition R9.1.2 [1] Let (X, τ) be a $T_{3\frac{1}{2}}$ space with normal basis $\mathcal{Z}$. Let $\mathcal{F}$ be a $\mathcal{Z}$-filter. $\mathcal{F}$ is prime if and only if $A \cup B \in \mathcal{F}$ with $A, B \in \mathcal{Z}$ implies $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

It is easy to check that every $\mathcal{Z}$-ultrafilter must be prime, and there is an example in [1] of a prime $\mathcal{Z}$-filter which is not a $\mathcal{Z}$-ultrafilter. However, the following does hold.

Lemma R9.1.3 Let (X, τ) be a $T_{3\frac{1}{2}}$ space with normal basis $\mathcal{Z}$, and let $\mathcal{F}$ be a prime $\mathcal{Z}$-filter. Then there is a unique $\mathcal{Z}$-ultrafilter containing $\mathcal{F}$. It is $\{Z \in \mathcal{Z} : Z \cap F \neq \emptyset$ for all $F \in \mathcal{F}\}$.

Proof: Let $\mathcal{G} = \{Z \in \mathcal{Z} : Z \cap F \neq \emptyset$ for all $F \in \mathcal{F}\}$. $\mathcal{G}$ is clearly a non-empty family of non-empty $\mathcal{Z}$-sets with the needed superset property. Let $Z_1, Z_2 \in \mathcal{G}$, let $F \in \mathcal{F}$, and suppose $Z_1 \cap Z_2 \cap F = \emptyset$. Since $\mathcal{Z}$ is a normal basis, there are $C, D \in \mathcal{Z}$ with $C \cup D = X$, $Z_1 \cap F \subseteq X - C$, and $Z_2 \cap F \subseteq X - D$. Since $\mathcal{F}$ is prime and contains X, either $C \cap F \in \mathcal{F}$ or $D \cap F \in \mathcal{F}$. But $Z_1 \cap (F \cap C)$ and $Z_2 \cap (F \cap D)$ are both empty, so that either case leads to a contradiction. Thus $\mathcal{G}$ is a $\mathcal{Z}$-filter. Clearly any $\mathcal{Z}$-filter containing $\mathcal{F}$ must be contained in $\mathcal{G}$, and so $\mathcal{G}$ is the unique $\mathcal{Z}$-ultrafilter containing $\mathcal{F}$.

Lemma R9.1.4 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Let $\mathcal{F} \in \omega(\mathcal{Z}_2)$. Then $\mathcal{F} \cap \mathcal{Z}_1$ is a prime $\mathcal{Z}_1$-filter.

Proof: This follows easily from the fact that the $\mathcal{Z}_2$-ultrafilter $\mathcal{F}$ must be prime.

Note that R9.1.3 and R9.1.4 allow the definition of a map $h : \omega(\mathcal{Z}_2) \to \omega(\mathcal{Z}_1)$ which reduces to the map used in the proof of R9.1.1iii if $\mathcal{F} \cap \mathcal{Z}_1$ is always a $\mathcal{Z}_1$-ultrafilter. At present I cannot determine whether the map must be continuous in the general case.

Lemma R9.1.5 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Assume that, whenever $A \in \mathcal{Z}_1$ and $B \in \mathcal{Z}_2$ with $A \cap B = \emptyset$, there is $C \in \mathcal{Z}_1$

with $B \subseteq C$ and $A \cap C = \emptyset$. Then for every $\mathcal{F}$ in $\omega(\mathcal{Z}_2)$, $\mathcal{F} \cap \mathcal{Z}_1$ is in $\omega(\mathcal{Z}_1)$.

Proof: Let $\mathcal{F}$ in $\omega(\mathcal{Z}_2)$ and suppose $A \in \mathcal{Z}_1$ with $A \cap F \neq \emptyset$ for all $F \in \mathcal{F} \cap \mathcal{Z}_1$. By the previous two lemmas, it is sufficient to show that $A \in \mathcal{F}$. If not, there is $B \in \mathcal{F}$ with $A \cap B = \emptyset$. By hypothesis there is $C \in \mathcal{Z}_1$ with $B \subseteq C$ and $A \cap C = \emptyset$. By the superset property for the $\mathcal{Z}_2$-filter $\mathcal{F}$, $C \in \mathcal{F} \cap \mathcal{Z}_1$, which contradicts the assumption about A.

Example R9.1.6 Let (X, τ) be T_4, let $\mathcal{Z}$ denote the collection of all closed subsets of X, and let $\mathcal{Z}(X)$ denote the zero-sets of X. By normality $\mathcal{Z}$ is a normal basis and of course the normal basis $\mathcal{Z}(X)$ is contained in $\mathcal{Z}$. Since disjoint closed sets can be separated by a continuous map into $[0, 1]$, the hypothesis of R9.1.5 holds. By R9.1.1iii $[(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})] \geq [(\omega(\mathcal{Z}(X)), \iota_{\mathcal{Z}(X)})]$. In fact, equality must hold since by P3.14 the latter is the class of the Stone-Čech compactification. For any X which is not perfectly normal, $\mathcal{Z}(X)$ is a proper subset of $\mathcal{Z}$, and so it is possible for distinct nested normal bases to generate equivalent compactifications.

For the rest of this section, definitions, results and terminology from subsection 3 of [8] will be used extensively.

Lemma R9.1.7 Let (X, τ) be an infinite discrete space and let E be an n-compatible equivalence relation on X. Then $\mathcal{Z}(E)$ is closed under complementaion.

Proof: Let $Z \in \mathcal{Z}(E)$ be associated with $\Delta \subseteq \{1, \ldots, n\}$. It is clear from the definitions that $X - Z$ is associated with $\{1, \ldots, n\} - \Delta$ and so $X - Z \in \mathcal{Z}(E)$.

Proposition R9.1.8 Let (X, τ) be an infinite discrete space. Let E and F be equivalence relations on X, which are respectively n-compatible and m-compatible. Assume $\mathcal{Z}(E) \subseteq \mathcal{Z}(F)$. Then $[(\omega(\mathcal{Z}(E)), \iota_{\mathcal{Z}(E)})] \leq [(\omega(\mathcal{Z}(F)), \iota_{\mathcal{Z}(F)})]$.

Proof: Let $A \in \mathcal{Z}(E)$ and $B \in \mathcal{Z}(F)$ with $A \cap B = \emptyset$. Then $B \subseteq X - A$ and $X - A \in \mathcal{Z}(E)$ by R9.1.7. Thus the hypothesis of R9.1.5 holds. The conclusion follows from that and R9.1.1iii.

Corollary R9.1.9 Let (X, τ) be an infinite discrete space. Let E and F be equivalence relations on X, which are respectively n-compatible and m-compatible.
Assume $\mathcal{Z}(E) \subseteq \mathcal{Z}(F)$. Then $n \leq m$.

Proof: The conclusion of the preceding result means there is a continuous surjection $h : \omega(\mathcal{Z}(F)) \to \omega(\mathcal{Z}(E))$ with $h \circ \iota_{\mathcal{Z}(F)} = \iota_{\mathcal{Z}(E)}$. Since h maps point filters to point filters, the preimages of the n non-point ultrafilters in $\omega(\mathcal{Z}(E))$ must lie in the set of m non-point ultrafilters of $\omega(\mathcal{Z}(F))$. Thus $n \leq m$.

Lemma R9.1.10 Let (X, τ) be an infinite discrete space. Let E and F be equivalence relations on X, which are respectively n-compatible and m-compatible. Let $E_1, \ldots, E_n$ and $F_1, \ldots, F_m$ be the distinct infinite equivalence classes of E and F respectively. Then $\mathcal{Z}(E) \subseteq \mathcal{Z}(F)$ if and only if $E_j \in \mathcal{Z}(F)$ for every j.

Proof: For $j \in \{1, \ldots, n\}$, E_j is associated with $\{1, \ldots, n\} - \{j\}$ and so $E_j \in \mathcal{Z}(E)$. Thus the desired conclusion is immediate if $\mathcal{Z}(E) \subseteq \mathcal{Z}(F)$. Now assume $E_j \in \mathcal{Z}(F)$ for all j, and, relative to F, let E_j be associated with $\gamma_j \subseteq \{1, \ldots, m\}$. This implies that $\cup\{F_i \cap (X - E_j) : i \notin \gamma_j\} = (\cup\{F_i : i \notin \gamma_j\}) - E_j$ is finite. Let $Z \in \mathcal{Z}(E)$ be associated (relative to E) with $\delta \subseteq \{1, \ldots, n\}$. For $j \in \delta$, since $Z \cap E_j$ is finite, $Z \cap (\cup\{F_i : i \notin \gamma_j\})$ is finite. Likewise, for $j \notin \delta$, since $(X - Z) \cap E_j$ is finite, $(X - Z) \cap (\cup\{F_i : i \notin \gamma_j\})$ is finite. Let $\Delta_1 = \cup\{\{1, \ldots, m\} - \gamma_j : j \in \delta\}$ and $\Delta_2 = \cup\{\{1, \ldots, m\} - \gamma_j : j \notin \delta\}$. Note that $Z \cap F_i$ is finite for $i \in \Delta_1$ and $(X - Z) \cap F_i$ is finite for $i \in \Delta_2$. I claim that Z is associated (relative to F) with Δ_1 and so $Z \in \mathcal{Z}(F)$. For the claim it is sufficient to show that $\Delta_2 = \{1, \ldots, m\} - \Delta_1$. Since each F_i is infinite, clearly $\Delta_1 \cap \Delta_2 = \emptyset$. Now suppose $i \in \{1, \ldots, m\}$ but $i \notin \Delta_1 \cup \Delta_2$. Then $i \in \gamma_j$ for every $j \in \{1, \ldots, n\}$ so that $F_i \cap (\cup_{j=1}^n E_j)$ is finite. Since $X - (\cup_{j=1}^n E_j)$ is also finite, F_i would have to be finite, a contradiction.

Lemma R9.1.11 Let (X, τ) be an infinite discrete space. Let F be an m-compatible equivalence relation on X, and let $F_1, \ldots, F_m$ be the distinct infinite equivalence classes of F. For each $i \in \{1, \ldots, m\}$, F_i^ω contains exactly one non-point ultrafilter.

Proof: By R5.3.7 the distinct non-point $\mathcal{Z}(F)$-ultrafilters are $\mathcal{G}_1, \ldots, \mathcal{G}_m$, where $\mathcal{G}_k = \{Z \in \mathcal{Z}(F) : Z \text{ is associated with some } \Delta \subseteq \{1, \ldots, m\} - \{k\}\}$. Since F_i is associated with $\{1, \ldots, m\} - \{i\}$, $F_i \in \mathcal{G}_i$ and $F_i \notin \mathcal{G}_k$ for $k \neq i$, i.e., $\mathcal{G}_i \in F_i^\omega$ and $\mathcal{G}_k \notin F_i^\omega$ for $k \neq i$.

The final result of this subsection establishes the converse of R9.1.8. To avoid ambiguity e and f will be used as sub- or superscripts to locate objects, e.g., $\mathcal{F}_x^e$ will denote the point ultrafilter of x in $\omega(\mathcal{Z}(E))$ and $(S^\omega)_f$ will denote the collection of $\mathcal{Z}(F)$-ultrafilters containing S.

Proposition R9.1.12 Let (X, τ) be an infinite discrete space. Let E and F be equivalence relations on X, which are respectively n-compatible and m-compatible.
If $[(\omega(\mathcal{Z}(E)), \iota_{\mathcal{Z}(E)})] \leq [(\omega(\mathcal{Z}(F)), \iota_{\mathcal{Z}(F)})]$, then $\mathcal{Z}(E) \subseteq \mathcal{Z}(F)$.

Proof: Let $E_1, \ldots, E_n$ and $F_1, \ldots, F_m$ be the distinct infinite equivalence classes of E and F respectively. By hypothesis there is $g : \omega(\mathcal{Z}(F)) \to \omega(\mathcal{Z}(E))$, a continuous surjection with $g(\mathcal{F}_x^f) = \mathcal{F}_x^e$ for all $x \in X$. Suppose $E_j \notin \mathcal{Z}(F)$. Then there is $i \in \{1, \ldots, m\}$ such that both $E_j \cap F_i$ and $(X - E_j) \cap F_i$ are infinite. Let $S_1 = \{\mathcal{F}_x^f : x \in E_j \cap F_i\}$ and $S_2 = \{\mathcal{F}_x^f : x \in (X - E_j) \cap F_i\}$. S_1 and S_2 are non-compact subsets of the closed set $(F_i^\omega)_f$. Then $c(S_1)$ and $c(S_2)$ (closures in $\omega(\mathcal{Z}(F))$) must both contain at least one non-point $\mathcal{Z}(F)$-ultrafilter and so by R9.1.11 $c(S_1) \cap c(S_2) \neq \emptyset$. But $S_1 \subseteq g^{-1}[(E_j^\omega)_e]$ and $S_2 \subseteq g^{-1}[(X - E_j)_e^\omega]$, which are disjoint closed sets, a contradiction. Thus $E_j \in \mathcal{Z}(F)$ for al j and the conclusion follows from R9.1.10.

Directed Sets and Suprema

Lemma R9.2.1 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ be a non-empty set of normal bases for X. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$. If $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ is a directed set under containment, then $\mathcal{Z}$ is a normal basis.

Proof: Since $\Gamma \neq \emptyset$, $\mathcal{Z}$ inherits the basis property and the disjunctive property (P3.1iii in [4]) from any $\mathcal{Z}_\gamma$. Because of the directed set assumption, any finite subset of $\mathcal{Z}$ must also be a subset of some $\mathcal{Z}_\gamma$. It follows immediately that $\mathcal{Z}$ satisfies the normality property (P3.1iv) and is closed under finite unions and intersections.

For the next three results of this subsection, the following will be assumed: (X, τ) is an infinite discrete space, and $\{E_\gamma : \gamma \in \Gamma\}$ is a non-empty set of equivalence relations on X with E_γ n_γ-compatible for each γ. Suppose $\{\mathcal{Z}(E_\gamma) : \gamma \in \Gamma\}$ is a directed set under containment, and let $\mathcal{Z} = \cup\{\mathcal{Z}(E_\gamma) : \gamma \in \Gamma\}$.

The notational convention for suprema described at the beginning of [7] will also be used.

In this context, R9.1.1ii implies that every point $\mathcal{Z}$-ultrafilter can be represented as a union, i.e., for any $x \in X$, $\mathcal{F}_x = \cup\{\mathcal{F}_x^\gamma : \gamma \in \Gamma\}$, where $\mathcal{F}_x^\gamma$ denotes the point filter determined by x in $\omega(\mathcal{Z}(E_\gamma)$. The following lemmas show that this representation extends to a general $\mathcal{Z}$-ultrafilter.

Lemma R9.2.2 If $\mathcal{F} \in \omega(\mathcal{Z})$, then $\mathcal{F} \cap \mathcal{Z}(E_\gamma) \in \omega(\mathcal{Z}(E_\gamma))$ for every $\gamma \in \Gamma$.

Proof: Let $\gamma \in \Gamma$, $A \in \mathcal{Z}_\gamma$, and $B \in \mathcal{Z}$ with $A \cap B = \emptyset$. Then $X - A \in \mathcal{Z}(E_\gamma)$ by R9.1.7 and $B \subseteq X - A$. The conclusion follows from R9.1.5.

The above says every $\mathcal{Z}$-ultrafilter can be described as a union of $\mathcal{Z}(E_\gamma)$-ultrafilters. The next lemma shows that $\mathcal{Z}$-ultrafilters can be constructed from below by taking a union of a suitable collection of $\mathcal{Z}(E_\gamma)$-ultrafilters.

Lemma R9.2.3 Let $\mathcal{F}_\gamma \in \omega(\mathcal{Z}(E_\gamma))$ for every $\gamma \in \Gamma$. Also assume that, for $\delta, \eta \in \Gamma$, if $\mathcal{Z}(E_\delta) \subseteq \mathcal{Z}(E_\eta)$, then $\mathcal{F}_\delta \subseteq \mathcal{F}_\eta$. Then $\mathcal{F} = \cup\{\mathcal{F}_\gamma : \gamma \in \Gamma\}$ is a $\mathcal{Z}$-ultrafilter.

Proof: Clearly $\mathcal{F}$ is a non-empty family of non-empty $\mathcal{Z}$-sets. Given F_1 and F_2 in $\mathcal{F}$, or $F \subseteq Z$ where $F \in \mathcal{F}$ and $Z \in \mathcal{Z}$, the hypothesis together with the directed set assumption for the normal bases imply that both sets are in $\mathcal{F}_\gamma$ for some γ. Thus $\mathcal{F}$ has the superset property and is closed under finite intersections. Now suppose $\mathcal{G}$ is a $\mathcal{Z}$-filter with $\mathcal{F} \subseteq \mathcal{G}$, and let $Z \in \mathcal{G}$. Pick γ with $Z \in \mathcal{Z}(E_\gamma)$. The $\mathcal{Z}(E_\gamma)$-ultrafilter $\mathcal{F}_\gamma$ is contained in the $\mathcal{Z}(E_\gamma)$-filter $\mathcal{G} \cap \mathcal{Z}(E_\gamma)$, and so $Z \in \mathcal{F}_\gamma \subseteq \mathcal{F}$. Thus $\mathcal{F}$ is a $\mathcal{Z}$-ultrafilter.

Proposition R9.2.4 $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is equivalent to $\bigvee\{(\omega(\mathcal{Z}(E_\gamma)), \iota_{\mathcal{Z}(E_\gamma)}) : \gamma \in \Gamma\}$.

Proof: By lemmas R9.2.2 and R9.1.1iii, $[(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})] \geq [(\omega(\mathcal{Z}(E_\gamma)), \iota_{\mathcal{Z}(E_\gamma)})]$ for every $\gamma \in \Gamma$. Now suppose $[(Y, f)] \geq [(\omega(\mathcal{Z}(E_\gamma)), \iota_{\mathcal{Z}(E_\gamma)}]$ for every $\gamma \in \Gamma$, and let $g_\gamma : Y \to \omega(\mathcal{Z}(E_\gamma))$ be the unique continuous surjection with $g_\gamma \circ f = \iota_{\mathcal{Z}(E_\gamma)}$. For $\delta, \eta \in \Gamma$ with $\mathcal{Z}(E_\delta) \subseteq \mathcal{Z}(E_\eta)$, by R9.1.8 there is a continuous surjection $h_{\eta\delta} : \omega(\mathcal{Z}(E_\eta)) \to \omega(\mathcal{Z}(E_\delta))$ such that $h_{\eta\delta} \circ \iota_{\mathcal{Z}(E_\eta)} = \iota_{\mathcal{Z}(E_\delta)}$. Combining the two functional equations, one obtains $h_{\eta\delta} \circ g_\eta \circ f = \iota_{\mathcal{Z}(E_\delta)}$ so that, by uniqueness, $h_{\eta\delta} \circ g_\eta = g_\delta$. Also, as in the proof of R9.1.1iii, $h_{\eta\delta}(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}(E_\delta)$. Thus for any $y \in Y$, $g_\eta(y) \cap \mathcal{Z}(E_\delta) = g_\delta(y)$. This shows that for any $y \in Y$ the hypothesis of R9.2.3 applies to $\{g_\gamma(y) : \gamma \in \Gamma\}$, and so a map $g : Y \to \omega(\mathcal{Z})$ can be defined by $g(y) = \cup\{g_\gamma(y) : \gamma \in \Gamma\}$. It is easy to check that $g \circ f = \iota_{\mathcal{Z}}$. To see that g is continuous, let $Z \in \mathcal{Z}$. It is sufficient to show that the inverse image of the basic closed set Z^ω is closed in Y. Pick $\eta \in \Gamma$ such that $Z \in \mathcal{Z}(E_\eta)$. Following the definitions, one easily sees that $g^{-1}[Z^\omega] = g_\eta^{-1}[Z_\eta^\omega]$, where Z_η^ω denotes the basic closed set of $\mathcal{Z}(E_\eta)$-ultrafilters containg Z. Since g_η is continuous, $g^{-1}[Z^\omega]$ is closed in Y. Since $g[Y]$ contains the dense $\iota_{\mathcal{Z}}[X]$ and g is closed because it is continuous from compact Y into a T_2 space, g is onto. Thus g is the map needed to show that $[(Y, f)] \geq [(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})]$.

Lemma R9.2.5 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, and let (Y, f) and (W, g) be finite-point compactifications of X. Then the supremum of (Y, f) and (W, g) is also a finite-point compactification of X. Moreover, if $|Y - f[X]| = k$ and $|W - g[X]| = n$, then the supremum has at most kn non-X points.

Proof: Let $S = \overline{\{(f(x), g(x)) : x \in X\}}$, where the closure is in $Y \times W$, and let $h : X \to S$ by $h(x) = (f(x), g(x))$. By R3.1.2 $(S, h) = (Y, f) \vee (W, g)$. Let $(a, b) \in S - h[X]$. There is a net $\{x_t\}$ such that $\{h(x_t)\} = \{(f(x_t), g(x_t))\}$ converges to (a, b). Then $f(x_t) \to a$ and $g(x_t) \to b$. If $a \in f[X]$ so that $a = f(x)$ for some x, then, since f is a homeomorphism from X to $f[X]$, $x_t \to x$ and so $g(x_t) \to g(x)$. By uniqueness of limits in a T_2 space, $g(x) = b$. Then $(a, b) = (f(x), g(x))$ is in $h[X]$, a contradiction. Thus $a \in Y - f[X]$ and similarly $b \in W - g[X]$. Therefore $S - h[X]$ is contaned in the finite set $(Y - f[X]) \times (W - g[X])$, which has at most kn points.

Theorem R9.2.6 Let (X, τ) be an infinite discrete space and let (Y, f) be a T_2 compactification of X. If (Y, f) can be represented as a supremum of finite-point compactifications, then there is a normal basis $\mathcal{Z}$ for X such that (Y, f) is equivalent to $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$.

Proof: Suppose $(Y, f) = \vee\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$, where $\Delta \neq \emptyset$ and each (Y_α, f_α) is a finite-point compactification of X. Let Γ be the set of non-empty finite subsets of Δ. For $\gamma \in \Gamma$ let $(S_\gamma, g_\gamma) = \vee\{(Y_\alpha, f_\alpha) : \alpha \in \gamma\}$. By R9.2.5 each (S_γ, g_γ) is a finite-point compactification and clearly $(Y, f) = \vee\{(S_\gamma, g_\gamma) : \gamma \in \Gamma\}$. By R5.1.2 and R5.3.8, for each $\gamma \in \Gamma$, there is an n_γ-compatible equivalence relation E_γ on X such that (S_γ, g_γ) is equivalent to $(\omega(\mathcal{Z}(E_\gamma)), \iota_{\mathcal{Z}(E_\gamma)})$. By R9.1.12, if $\delta, \eta \in \Gamma$ and $\delta \subseteq \eta$, then $\mathcal{Z}(E_\delta) \subseteq \mathcal{Z}(E_\eta)$. Thus $\{\mathcal{Z}(E_\gamma) : \gamma \in \Gamma\}$ is a directed set under containment and so the hypothses of R9.2.4 are satisfied. $\mathcal{Z} = \cup\{\mathcal{Z}(E_\gamma) : \gamma \in \Gamma\}$ is the required normal basis.

Connection to the Frink Question

In 1964 Frink [3] asked whether every compactification of a $T_{3\frac{1}{2}}$ space (X, τ) must be equivalent to $\omega(\mathcal{Z})$ for some normal basis $\mathcal{Z}$ of X. In Theorem 8.11 of [2], Chandler shows that Frink's question can be reduced to the discrete case, i.e., the answer is positive in general if it is positive for every compactification of an infinite discrete space. Theorem R9.2.6 above identifies a class of compactifications of discrete spaces which can be generated with a normal basis. This subsection will examine that class in more detail.

Proposition R9.3.1 Let (X, τ) be a non-compact, locally compact, extremely disconnected T_2 space and let (Y, f) be a finite-point T_2 compactification of X. Then Y is zero-dimensional.

Proof: Let $|Y - f[X]| = n$. By R5.1.2 [8] there is an n-star of X, which determines an n-point compactification equivalent to (Y, f), i.e., there is a pairwise disjoint family $\{G_i : i = 1, \ldots, n\}$ of open sets in X whose union has a compact complement K such that $K \cup G_i$ is not compact for each i. The equivalent compactification is (W, h), where $W = X \cup \{p_1, \ldots, p_n\}$ for some distinct $p_1, \ldots, p_n$ not in X, $h : X \to W$ by $h(x) = x$, and the topology on W is $\sigma = \{O \subseteq W : O \cap X \in \tau$ and $p_i \in O \Rightarrow (X - O) \cap G_i$ has compact closure in $X\}$. It is sufficient to show that (W, σ) has a basis of clopen sets.

Let $O \in \sigma$ and first consider $x \in O \cap X$. Since X is extremely disconnected and regular, X is zero-dimensional. Using local compactness, there is F compact and clopen in X such that $x \in F \subseteq O \cap X$. Clearly $F \in \sigma$ and, since F is compact as a subset of W, F is clopen in W. Now suppose $p_i \in O$. Since $G_i \cup \{p_i\} \in \sigma$, by regularity there is $G^* \in \sigma$ such that $p_i \in G^* \subseteq c_W(G^*) \subseteq O \cap (G_i \cup \{p_i\})$. Let $G = G^* \cap X$. Since X is extremely disconnected and $G \in \tau$, $c_X(G)$ is clopen in X. Since G is a subset of the X-closed $c_W(G^*) \cap X$, we have $c_X(G) \subseteq c_W(G^*)$ and so $p_i \in c_X(G) \cup \{p_i\} \subseteq c_W(G^*) \subseteq O$. To finish, it is sufficient to show that $c_X(G) \cup \{p_i\}$ is clopen in W. First, $(c_X(G) \cup \{p_i\}) \cap X = c_X(G)$, which is open in X. Also $(X - (c_X(G) \cup \{p_i\})) \cap G_i \subseteq (X - G^*) \cap G_i$, which has compact closure in X. Thus $c_X(G) \cup \{p_i\} \in \sigma$. Finally, let $H = W - (c_X(G) \cup \{p_i\})$. $H \cap X = X - c_X(G) \in \tau$. For $j \neq i$, $p_j \in H$ and $(X - H) \cap G_j = c_X(G) \cap G_j = \emptyset$, which is compact. Thus $H \in \sigma$ and $c_X(G) \cup \{p_i\}$ is also closed in W.

Corollary R9.3.2 Let (X, τ) be an infinite discrete space and let (Y, f) be a finite-point T_2 compactification of X. Then Y is zero-dimensional.

Proof: An infinite discrete space satifies the hypothesis of R9.3.1.

For the rest of this subsection some notation from [5] will be used: $\mathcal{TB}(X)$ denotes the set of totally bounded uniformities on X that generate τ; for $\mathcal{U} \in \mathcal{TB}(X)$, $\Psi_0(\mathcal{U})$ denotes the equivalence class of compactifications associated with $\mathcal{U}$; and, for locally compact X,

$\mathcal{U}_m$ denotes the smallest element of $\mathcal{TB}(X)$. When X is discrete, $\mathcal{U}_m = \{U \subseteq X \times X : U \supseteq \cup_{i=1}^n S_i \times S_i$ where $S_1, \ldots S_n$ covers X and at least one S_i has a finite complement $\}$.

Theorem R9.3.3 Let (X, τ) be an infinite discrete space and let (Y, f) be a T_2 compactification of X. Then Y is zero-dimensional if and only if (Y, f) can be represented as a supremum of finite-point compactifications.

Proof: By the Magill-Glasenapp theorem (R6.3.11 in [9]), the supremum of a collection of zero-dimensional compactifications is also zero-dimensional. This combined with R9.3.2 verifies one direction of the conclusion.

For the converse assume Y is zero-dimensional and let $\mathcal{U} \in \mathcal{TB}(X)$ with $\Psi_0(\mathcal{U}) = [(Y, f)]$. By R1.5 [5] and R5.2.4 [8] it is sufficient to show that $\mathcal{U} = \bigvee\{\mathcal{U}_m \vee \mathcal{U}_E : E$ is an n-compatible equivalence relation on X for some n and $E \in \mathcal{U}\}$. It is clear that $\mathcal{U}$ contains the supremum, since $\mathcal{U}_m \subseteq \mathcal{U}$. For the reverse containment let $U \in \mathcal{U}$ and let $\mathcal{V}$ be the unique uniformity for Y. The map f from $(X, \mathcal{U})$ is a unimorphism onto $f[X]$, where $f[X]$ has the subspace uniformity from Y. Thus let $V \in \mathcal{V}$ such that $f \times f[U] = (f[X] \times f[X]) \cap V$, and let $W \in \mathcal{V}$ with $W = W^{-1}$ and $W \circ W \subseteq V$. It follows easily that $W[y] \times W[y] \subseteq V$ for any $y \in Y$. By the zero-dimensionality of Y, for each $y \in Y$ there is a clopen G_y with $y \in G_y \subseteq W[y]$ and $G_y \times G_y \subseteq V$. By compactness there is a finite subcover $G_1, \ldots, G_m$. Let $H_1 = G_1$ and for $2 \leq j \leq m$ let $H_j = G_j - (\cup_{i=1}^{j-1} G_i)$. Each H_i is clopen and $H_1, \ldots, H_m$ is a partition of Y. The corresponding equivalence relation $F = \cup_{i=1}^m H_i \times H_i$ is an open neighborhood of the diagonal in Y and so $F \in \mathcal{V}$. Moreover, since $H_i \times H_i \subseteq G_i \times G_i$, $F \subseteq V$. Let $E = (f \times f)^{-1}[(f[X] \times f[X]) \cap F]$. Then $E \in \mathcal{U}$ and $E \subseteq U$. It is routine to verify that E is an n-compatible equivalence relation on X for some $n \leq m$ and so $U \in \mathcal{U}_m \vee \mathcal{U}_E \subseteq \mathcal{U}$, i.e. U is in the supremum.

Corollary R9.3.4 Let (X, τ) be an infinite discrete space and let (Y, f) be a T_2 compactification of X. If Y is zero-dimensional, then there is a normal basis $\mathcal{Z}$ for X such that (Y, f) is equivalent to $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$.

Proof: This is immediate from R9.3.3 and R9.2.6.

This subsection concludes with an example: a compactification of a discrete space which is not zero-dimensional. The following lemma, which applies to any infinite discrete space, will be used below.

Lemma R9.3.5 Let (X, τ) be a non-compact, locally compact T_2 space. Let E, F be equivalence relations on X with E k-compatible and F l-compatible. Assume that the equivalence classes of both E and F are all open. Then $E \cap F$ is an n-compatible equivalence relation for some $n \leq kl$. In addition, all the equivalence classes of $E \cap F$ are open.

Proof: Clearly, $E \cap F$ is an equivalence relation for X with equivalence classes given by $[x]_{E \cap F} = [x]_E \cap [x]_F$ so that the equivalence classes of $E \cap F$ are all open. Let $G_1, \ldots, G_k$ be the E-equivalence classes which form a k-star and let $H_1, \ldots, H_l$ be the F-classes which form an l-star. There are kl open sets of the form $G_i \cap H_j$, which form a pairwise disjoint collection. Let $K_1 = X - \cup_{i=1}^k G_i$ and $K_2 = X - \cup_{j=1}^l H_j$. Then $K = X - \cup\{G_i \cap H_j : 1 \leq i \leq k$ and $1 \leq j \leq l\} = K_1 \cup K_2$, which is compact. Let n be the number of $G_i \cap H_j$ such that $K \cup (G_i \cap H_j)$ is non-compact. By definition these form an n-star and so $E \cap F$ is n-compatible.

For rest of this subsection (X, τ) denotes $[0, 1]$ with the discrete topology and $\mathcal{V}$ the uniformity for X generated by the absolute value metric. As mentioned in [4], basic sets for $\mathcal{V}$ are of the form $V_\epsilon = \{(x,y) : |x-y| < \epsilon\}$. Exponents will indicate iterated compsition of a relation, e.g., $R^2 = R \circ R$, $R^3 = R \circ R \circ R$, etc.

Lemma R9.3.6 Let γ and δ be positive real numbers, and let S be a subset of X with $X - S$ finite. Let $W = (S \times S) \cup \{(x,x) : x \in X - S\}$. Then $(V_\gamma \cap W) \circ (V_\delta \cap W) = V_{\gamma+\delta} \cap W$.

Proof: By the triangle inequality $V_\gamma \circ V_\delta \subseteq V_{\gamma+\delta}$, which immediately yields $(V_\gamma \cap W) \circ (V_\delta \cap W) \subseteq V_{\gamma+\delta} \cap W$. For the reverse containment let $(x,y) \in V_{\gamma+\delta} \cap W$ and assume without loss of generality that $x \le y$. If $y < x + \gamma$, then $(x,y) \in V_\gamma \cap W \subseteq (V_\gamma \cap W) \circ (V_\delta \cap W)$. Otherwise $x + \gamma \le y < x + \gamma + \delta$. Let $\epsilon = x + \gamma + \delta - y$. Since $X - S$ is finite, it is possible to pick $t \in S$ with $\max\{x, x + \gamma - \epsilon\} < t < x + \gamma$. Then $(x,t) \in V_\gamma \cap W$ and $|y - t| < \delta$ so that $(y,t) \in V_\delta \cap W$. Therefore $(x,y) \in (V_\gamma \cap W) \circ (V_\delta \cap W)$.

Example R9.3.7 Let $\mathcal{U} = \mathcal{V} \vee \mathcal{U}_m$, where as usual $\mathcal{U}_m$ denotes the smallest element of $\mathcal{TB}((X,\tau))$. Since $(X,\mathcal{V})$ is totally bounded, $\mathcal{U} \in \mathcal{TB}((X,\tau))$ by P2.13 and P2.14 [4]. It will be shown that the compactification $\Psi_0(\mathcal{U})$ is not zero-dimensional. By R9.3.3 it is sufficient to show that $\Psi_0(\mathcal{U})$ is not the supremum of finite-point compactifications or, equivalently, that $\mathcal{U} \ne \bigvee\{\mathcal{U}_m \vee \mathcal{U}_E : E$ is an n-compatible equivalence relation on X for some n and $E \in \mathcal{U}\}$. To be specific, it will be shown that $V_{\frac{1}{2}}$, which is in $\mathcal{U}$, is not in the supremum. If we deny this, then, using R9.3.5 and the fact that uniformities are closed under finite intersections, there must be an n-compatible equivalence relation $E \in \mathcal{U}$ and $W_1 \in \mathcal{U}_m$ such that $E \cap W_1 \subseteq V_{\frac{1}{2}}$. By the definition of $\mathcal{U}_m$ there must be S_1 such that $S_1 \times S_1 \subseteq W_1$ and $X - S_1$ is finite. Let $V = \{(x,x) : x \in X - S_1\} \cup S_1 \times S_1$. $V \in \mathcal{U}_m$ and so $V \cap E \in \mathcal{U}$. There is $W_2 \in \mathcal{U}_m$ and $\epsilon > 0$ such that $W_2 \cap V_\epsilon \subseteq V \cap E$. As before, there is S_2 such that $S_2 \times S_2 \subseteq W_2$ and $X - S_2$ is finite. Let $W = \{(x,x) : x \in X - S_2\} \cup S_2 \times S_2$. Then $W \cap V_\epsilon \in \mathcal{U}$ and $W \cap V_\epsilon \subseteq V \cap E \subseteq V_{\frac{1}{2}}$. Since $V \cap E$ is an equivalence relation, for any n, $(V \cap E)^n = V \cap E$. From these facts and R9.3.6 $(W \cap V_\epsilon)^n = W \cap V_{n\epsilon} \subseteq V_{\frac{1}{2}}$ for all n. Pick n large enough that $n\epsilon > 1$. Pick $t \in [0, \frac{1}{5}] \cap S_2$ and $s \in [\frac{4}{5}, 1] \cap S_2$. Then $(s,t) \in W \cap V_{n\epsilon}$ but $(s,t) \notin V_{\frac{1}{2}}$, a contradiction.

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http://www.susanjkleinart.com/compactification/

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8. This website, R5: Finite Point Compactifications

9. This website, R6: Suprema of Two-point Compactifications

Added Comments 2007

Chandler and Faulkner [10] state that the Frink question was resolved by Ul'yanov [11], who proved that "the assertion that every Hausdorff bicompact extension of an arbitrary separable completely regular space is an extension of Wallman type is equivalent to the continuum hypothesis."

Additional References

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Added 2023

In [12] it is shown how a normal basis $\mathcal{Z}$ determines the separated totally bounded uniformity associated with the class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. This allows the removal of certain restrictions in R9.1.1.iii and R9.2.4 as follows:

Corollary R14.Add.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Then $(\omega(\mathcal{Z}_1), \iota_{\mathcal{Z}_1}) \leq (\omega(\mathcal{Z}_2), \iota_{\mathcal{Z}_2})$.

Corollary R14.Add.7 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\{\mathcal{Z}_\delta : \delta \in \Delta\}$ be a non-empty collection of normal bases for (X, τ) which is a directed set relative to containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\delta : \delta \in \Delta\}$. Then $\mathcal{Z}$ is a normal basis for (X, τ) and $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ acts as a supremum for $\{(\mathcal{Z}_\delta, \iota_{\mathcal{Z}_\delta}) : \delta \in \Delta\}$.

Given normal bases $\mathcal{Z}_1$ and $\mathcal{Z}_2$ for (X, τ) with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, and knowing R14.Add.2, one can describe the connecting continuous surjection in the definition of $\leq$. This uses the facts that $\mathcal{F}$ in $\omega(\mathcal{Z})$ implies $\mathcal{F} \cap \mathcal{Z}_1$ is a prime $\mathcal{Z}_1$-filter (R9.1.4) and that a prime $\mathcal{Z}_1$-filter is contained in a unique $\mathcal{Z}_1$-ultrafilter (R9.1.3).

The notation for the next result is complicated by considering two normal bases. The embeddings map $x \in X$ to the respective point ultrafilters; in the notation from the beginning of this section, ${\mathcal{F}_x}^2$ in $\omega(\mathcal{Z}_2)$ and ${\mathcal{F}_x}^1$ in $\omega(\mathcal{Z}_1)$. For $Z \in \mathcal{Z}_i$, Z^{ω_i} is the set of all elements of $\omega(\mathcal{Z}_i)$ containing Z. By P3.6 the collection of all Z^{ω_i} for $Z \in \mathcal{Z}_i$ is a normal basis for $\omega(\mathcal{Z}_i)$.

Proposition R9.Add.1 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Let $\phi : \omega(\mathcal{Z}_2) \to \omega(\mathcal{Z}_1)$ be the continuous map such that $\phi \circ \iota_{\mathcal{Z}_2} = \iota_{\mathcal{Z}_1}$. Then for $\mathcal{F} \in \omega(\mathcal{Z}_2)$, $\phi(\mathcal{F})$ is the unique $\mathcal{Z}_1$-ultrafilter containing $\mathcal{F} \cap \mathcal{Z}_1$.

Proof: Let $\mathcal{F} \in \omega(\mathcal{Z}_2)$. By R9.1.3 and R9.1.4, since $\phi(\mathcal{F})$ is a $\mathcal{Z}_1$-ultrafilter, it is sufficient to show that $\mathcal{F} \cap \mathcal{Z}_1 \subseteq \phi(\mathcal{F})$. Let $Z \in \mathcal{F} \cap \mathcal{Z}_1$ and suppose $Z \notin \phi(\mathcal{F})$. By P3.3ii there is $W \in \phi(\mathcal{F}) \subseteq \mathcal{Z}_1$ such that $Z \cap W = \emptyset$. By the definition of a normal basis, there are $C, D \in \mathcal{Z}_1$ such that $C \cup D = X$, $Z \subseteq X - C$, and $W \subseteq X - D$. Let $\{x_\alpha\}$ be a net in X such that $\iota_{\mathcal{Z}_2}(x_\alpha) = {\mathcal{F}_{x_\alpha}}^2$ converges to $\mathcal{F}$. By the continuity of ϕ and the hypothesis, $\phi({\mathcal{F}_{x_\alpha}}^2) = \iota_{\mathcal{Z}_1}(x_\alpha) = {\mathcal{F}_{x_\alpha}}^1$ converges to $\phi(\mathcal{F})$. Since $D \cap W = \emptyset$, $D \notin \phi(\mathcal{F})$, i.e., $\phi(\mathcal{F})$ is in the open set $\omega(\mathcal{Z}_1) - D^{\omega_1}$. By convergence there is α_0 such that $\alpha \geq \alpha_0$ implies ${\mathcal{F}_{x_\alpha}}^1$ is in that open set, i.e., $D \notin {\mathcal{F}_{x_\alpha}}^1$, i.e., $x_\alpha \notin D$. Since $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, both C and Z are in $\mathcal{Z}_2$. Since

$Z \in \mathcal{F}$ and $C \cap Z = \emptyset$, $C \notin \mathcal{F}$, i.e., $\mathcal{F}$ is in the open set $\omega(\mathcal{Z}_2) - C^{\omega_2}$. By convergence there is α_1 such that $\alpha \geq \alpha_1$ implies $\mathcal{F}_{x_\alpha}^{\ 2}$ is in the open set, i.e. $C \notin \mathcal{F}_{x_\alpha}^{\ 2}$, i.e., $x_\alpha \notin C$. By the directed set property of the indices there is γ with $\gamma \geq \alpha_0$ and $\gamma \geq \alpha_1$. Then $x_\gamma \notin D$ and $x_\gamma \notin C$, which contradicts $x_\gamma \in X = C \cup D$.

The conclusion of the previous result suggests a question: Could one define the map ϕ first and then show that it is the continuous connecting function needed for the relation? The next two results give a positive answer, which would yield R14.Add.2 and eventually R14.Add.7 without using the uniformity as described in [12].

Lemma R9.Add.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Let $\phi : \omega(\mathcal{Z}_2) \to \omega(\mathcal{Z}_1)$ be defined for $\mathcal{F} \in \omega(\mathcal{Z}_2)$ by letting $\phi(\mathcal{F})$ be the unique $\mathcal{Z}_1$-ultrafilter containing $\mathcal{F} \cap \mathcal{Z}_1$. Let $\mathcal{G} \in \omega(\mathcal{Z}_2)$ and let $\{x_\alpha\}$ be a net in X such that $\{\mathcal{F}_{x_\alpha}^{\ 2}\}$ converges to $\mathcal{G}$. Then $\{\mathcal{F}_{x_\alpha}^{\ 1}\}$ converges to $\phi(\mathcal{G})$.

Proof: Let O be open in $\omega(\mathcal{Z}_1)$ with $\phi(\mathcal{G})$ in O. By P3.6 there is $Z \in \mathcal{Z}_1$ such that $\omega(\mathcal{Z}_1) - O \subseteq Z^{\omega_1}$ and $\phi(\mathcal{G}) \notin Z^{\omega_1}$, i.e., $Z \notin \phi(\mathcal{G})$. Suppose $\mathcal{G} \in Z^{\omega_2}$, i.e., $Z \in \mathcal{G}$. Then $Z \in \mathcal{G} \cap \mathcal{Z}_1$, which is contained in $\phi(\mathcal{G})$ by definition, and so $Z \in \phi(\mathcal{G})$, a contradiction. Thus $\mathcal{G}$ is not in the $\omega(\mathcal{Z}_2)$-closed set Z^{ω_2}. By the given convergence there is α_0 such that $\alpha \geq \alpha_0$ implies $\mathcal{F}_{x_\alpha}^{\ 2} \notin Z^{\omega_2}$, i.e., $Z \notin \mathcal{F}_{x_\alpha}^{\ 2}$, i.e, $x_\alpha \notin Z$ by the definition of a point-filter. Then $Z \notin \mathcal{F}_{x_\alpha}^{\ 1}$, i.e., $\mathcal{F}_{x_\alpha}^{\ 1} \notin Z^{\omega_1}$, i.e., $\mathcal{F}_{x_\alpha}^{\ 1} \in \omega(\mathcal{Z}_1) - Z^{\omega_1}$, which is contained in O. The conclusion follows by the definition of convergence.

The next result, the proof of which bears a clear similarity to the proof of R9.Add.1, uses the lemma as a substitute for the hypothesis of continuity.

Proposition R9.Add.3 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Let $\phi : \omega(\mathcal{Z}_2) \to \omega(\mathcal{Z}_1)$ be defined for $\mathcal{F} \in \omega(\mathcal{Z}_2)$ by letting $\phi(\mathcal{F})$ be the unique $\mathcal{Z}_1$-ultrafilter containing $\mathcal{F} \cap \mathcal{Z}_1$. Then

i) $\phi \circ \iota_{\mathcal{Z}_2} = \iota_{\mathcal{Z}_1}$.

ii) ϕ is continuous.

Proof: The embeddings map $x \in X$ to the respective point ultrafilters, i. e., $\mathcal{F}_x^{\ 2}$ in $\omega(\mathcal{Z}_2)$ and $\mathcal{F}_x^{\ 1}$ in $\omega(\mathcal{Z}_1)$. By R9.1.1ii $\mathcal{F}_x^{\ 2} \cap \mathcal{Z}_1$ is the $\mathcal{Z}_1$-ultrafilter $\mathcal{F}_x^{\ 1}$ and so by definition $\phi(\mathcal{F}_x^{\ 2}) = \mathcal{F}_x^{\ 1}$. That equation expressed in terms of the embeddings is i). For part ii): Because the set of all Z^{ω_1} over all $Z \in \mathcal{Z}_1$ is a closed base for $\omega(\mathcal{Z}_1)$, it is sufficient to show that, for every $Z \in \mathcal{Z}_1$, $\phi^{-1}[Z^{\omega_1}]$ is closed in $\omega(\mathcal{Z}_2)$. Let $Z \in \mathcal{Z}_1$ and suppose $\mathcal{G} \notin \phi^{-1}[Z^{\omega_1}]$. Then $Z \notin \phi(\mathcal{G})$ and so, by the definition of ϕ, there is $W \in \mathcal{G} \cap \mathcal{Z}_1$ with $W \cap Z = \emptyset$. By the definition of a normal basis there are C, D in $\mathcal{Z}_1$ such that $C \cup D = X$, $Z \subseteq X - C$, and $W \subseteq X - D$. $W \cap D = \emptyset$ implies $D \notin \mathcal{G}$, i.e., $\mathcal{G} \notin D^{\omega_2}$. To finish, it is sufficient to show that $\phi^{-1}[Z^{\omega_1}] \subseteq D^{\omega_2}$. Let $\mathcal{H} \in \phi^{-1}[Z^{\omega_1}]$ and suppose $\mathcal{H} \notin D^{\omega_2}$, i.e., $\mathcal{H}$ is in the open set $\omega(\mathcal{Z}_2) - D^{\omega_2}$. Let $\{x_\alpha\}$ be a net in X such that $\{\mathcal{F}_{x_\alpha}^{\ 2}\}$ converges to $\mathcal{H}$. Then there is α_0 such that $\alpha \geq \alpha_0$ implies $\mathcal{F}_{x_\alpha}^{\ 2} \notin D^{\omega_2}$, i.e., $D \notin \mathcal{F}_{x_\alpha}^{\ 2}$, i.e. $x_\alpha \notin D$. By the previous lemma, $\{\mathcal{F}_{x_\alpha}^{\ 1}\}$ converges to $\phi(\mathcal{H})$. Since $Z \in \phi(\mathcal{H})$ and $Z \cap C = \emptyset$, $C \notin \phi(\mathcal{H})$ and so $\phi(\mathcal{H})$ is in the open set $\omega(\mathcal{Z}_1) - C^{\omega_1}$. By convergence there is α_1 such that $\alpha \geq \alpha_1$ implies $\mathcal{F}_{x_\alpha}^{\ 1} \notin C^{\omega_1}$, i.e., $C \notin \mathcal{F}_{x_\alpha}^{\ 1}$, i.e. $x_\alpha \notin C$. By the directed set property there is γ with $\gamma \geq \alpha_0$ and $\gamma \geq \alpha_1$. Then $x_\gamma \in X$, $x_\gamma \notin C$, and $x_\gamma \notin D$. But that contradicts $C \cup D = X$.

R9.2.4 was proven in a limited context, i.e., a discrete space with a directed set of normal bases generated by n-compatible equivalence relations. The next two results show that R9.Add.3 makes possible a partial generalization of R9.2.4. Recall that $\mathcal{Z}$ in the next two results is a normal basis by R9.2.1.

Lemma R9.Add.4 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ be a non-empty set of normal bases for X, with $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ a directed set under containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$. Assume, $\mathcal{F}_\alpha \in \omega(\mathcal{Z}_\alpha)$ for every $\alpha \in \Gamma$. Finally assume, for every $\alpha, \beta \in \Gamma$, that $\mathcal{Z}_\alpha \subseteq \mathcal{Z}_\beta$ implies $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$. Then $\cup\{\mathcal{F}_\alpha : \alpha \in \Gamma\}$ is a $\mathcal{Z}$-ultrafilter.

Proof: Let $\mathcal{F} = \cup\{\mathcal{F}_\alpha : \alpha \in \Gamma\}$. Since $\Gamma \neq \emptyset$, by definition $\mathcal{F}$ is a non-empty collection of non-empty $\mathcal{Z}$-sets. Let Z, W be in $\mathcal{F}$ with $Z \in \mathcal{F}_\alpha$ and $W \in \mathcal{F}_\beta$. By hypothesis there is γ with $\mathcal{Z}_\alpha \cup \mathcal{Z}_\beta \subseteq \mathcal{Z}_\gamma$ so that $\mathcal{F}_\alpha \cup \mathcal{F}_\beta \subseteq \mathcal{F}_\gamma$. Then $Z \cap W \in \mathcal{F}_\gamma \subseteq \mathcal{F}$. Similarly, for $Z \in \mathcal{F}$ and $W \in \mathcal{Z}$ with $Z \subseteq W$, $W \in \mathcal{F}$. Thus $\mathcal{F}$ is a $\mathcal{Z}$-filter. Now let $\mathcal{G}$ be a $\mathcal{Z}$-filter with $\mathcal{F} \subseteq \mathcal{G}$. Let $\alpha \in \Gamma$. By R9.1.1i $\mathcal{G} \cap \mathcal{Z}_\alpha$ is a $\mathcal{Z}_\alpha$-filter. Since $\mathcal{F}_\alpha \subseteq (\mathcal{F} \cap \mathcal{Z}_\alpha) \subseteq (\mathcal{G} \cap \mathcal{Z}_\alpha)$ and $\mathcal{F}_\alpha$ is a $\mathcal{Z}_\alpha$-ultrafilter, $\mathcal{F}_\alpha = \mathcal{G} \cap \mathcal{Z}_\alpha$. Since $\mathcal{G} = \cup\{\mathcal{G} \cap \mathcal{Z}_\alpha : \alpha \in \Gamma\}$, $\mathcal{G} = \mathcal{F}$ and the conclusion holds.

Proposition R9.Add.5 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ be a non-empty set of normal bases for X, with $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ a directed set under containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$. Assume, for every $\alpha, \beta \in \Gamma$ with $\mathcal{Z}_\alpha \subseteq \mathcal{Z}_\beta$, $\mathcal{G} \in \omega(\mathcal{Z}_\beta)$ implies $\mathcal{G} \cap \mathcal{Z}_\alpha \in \omega(\mathcal{Z}_\alpha)$. Then $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ acts as the supremum of $\{(\omega(\mathcal{Z}_\alpha), \iota_{\mathcal{Z}_\alpha}) : \alpha \in \Gamma\}$.

Proof: By R9.Add.3 (or R14.Add.2) $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is an upper bound of $\{(\omega(\mathcal{Z}_\alpha), \iota_{\mathcal{Z}_\alpha}) : \alpha \in \Gamma\}$. Now let (Y, f) be an upper bound of $\{(\omega(\mathcal{Z}_\alpha), \iota_{\mathcal{Z}_\alpha}) : \alpha \in \Gamma\}$. For $\alpha \in \Gamma$ let $\phi_\alpha : Y \to \omega(\mathcal{Z}_\alpha)$ be the continuous map with $\phi_\alpha \circ f = \iota_{\mathcal{Z}_\alpha}$. For $\alpha, \beta \in \Gamma$ with $\mathcal{Z}_\alpha \subseteq \mathcal{Z}_\beta$, let $g_{\beta\alpha} : \omega(\mathcal{Z}_\beta) \to \omega(\mathcal{Z}_\alpha)$ be the continuous map with $g_{\beta\alpha} \circ \iota_{\mathcal{Z}_\beta} = \iota_{\mathcal{Z}_\alpha}$. By R9.Add.3 and the hypothesis, $g_{\beta\alpha}(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_\alpha$ for $\mathcal{F} \in \omega(\mathcal{Z}_\beta)$. Because the connecting maps are unique $g_{\beta\alpha} \circ \phi_\beta = \phi_\alpha$. Now let $y \in Y$. For $\alpha, \beta \in \Gamma$ with $\mathcal{Z}_\alpha \subseteq \mathcal{Z}_\beta$, these equations show that $\phi_\alpha(y) = \phi_\beta(y) \cap \mathcal{Z}_\alpha$ so that $\phi_\alpha(y) \subseteq \phi_\beta(y)$. By R9.Add.4 $\psi : Y \to \omega(\mathcal{Z})$ can be defined by $\psi(y) = \cup\{\phi_\alpha(y) : \alpha \in \Gamma\}$. For $x \in X$, $\phi_\alpha(f(x))$ is the $\mathcal{Z}_\alpha$-point filter of x. It follows easily that $\psi(f(x))$ is the $\mathcal{Z}$-point filter of x, i.e., $\psi \circ f = \iota_{\mathcal{Z}}$. Next it will be shown that ψ is continuous. Let $Z \in \mathcal{Z}$. By P3.6 it is sufficient to verify $\psi^{-1}[Z^\omega]$ is closed in Y. There is $\alpha \in \Gamma$ such that $Z \in \mathcal{Z}_\alpha$. It is claimed that $\psi^{-1}[Z^\omega] = \phi_\alpha^{-1}[Z^{\omega_\alpha}]$, where the ω_α superscript indicates the omega operation relative to $\mathcal{Z}_\alpha$. Let $y \in \phi_\alpha^{-1}[Z^{\omega_\alpha}]$, i.e., $\phi_\alpha(y) \in Z^{\omega_\alpha}$, i.e., $Z \in \phi_\alpha(y)$. By definition $Z \in \psi(y)$ so that $y \in \psi^{-1}[Z^\omega]$. Conversely, let $y \in \psi^{-1}[Z^\omega]$, i.e., $Z \in \psi(y)$. By definition there is $\beta \in \Gamma$ with $Z \in \phi_\beta(y)$. By the directed set property there is $\gamma \in \Gamma$ such that $\mathcal{Z}_\alpha \cup \mathcal{Z}_\beta \subseteq \mathcal{Z}_\gamma$. As above, $\phi_\beta(y) \subseteq \phi_\gamma(y)$ and $\phi_\alpha(y) = \phi_\gamma(y) \cap \mathcal{Z}_\alpha$. Because $Z \in \phi_\gamma(y)$ and $Z \in \mathcal{Z}_\alpha$, $Z \in \phi_\alpha(y)$, i.e., $y \in \phi_\alpha^{-1}[Z^{\omega_\alpha}]$ and the claim is verified. Since ϕ_α is continuous and Z^{ω_α} is closed in $\omega(\mathcal{Z}_\alpha)$, $\phi_\alpha^{-1}[Z^{\omega_\alpha}]$, and so $\psi^{-1}[Z^\omega]$, is closed in Y as required. Thus $(Y, f) \geq (\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ and so $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is the least upper bound, i.e., the conclusion holds.

R9.Add.3 could be simplified and R9.Add.5 improved if one could show that, with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, $\mathcal{F} \cap \mathcal{Z}_1$ is a $\mathcal{Z}_1$-ultrafilter when $\mathcal{F} \in \omega(\mathcal{Z}_2)$, i.e., not just prime as shown in R9.1.4. A proof so far eludes me, as does a counterexample. The rest of this added subsection provides partial results and examples related to this issue.

Lemma R9.Add.6 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Assume $(\omega(\mathcal{Z}_2), \iota_{\mathcal{Z}_2})$ is equivalent to $(\omega(\mathcal{Z}_1), \iota_{\mathcal{Z}_1})$. Then for every $\mathcal{F} \in \omega(\mathcal{Z}_2)$, $\mathcal{F} \cap \mathcal{Z}_1$ is a $\mathcal{Z}_1$-ultrafilter.

Proof: Let ϕ be the map described in R9.Add.3. Because the connecting map for the relation is unique, the hypothesis of equivalence implies that ϕ is a homeomorphism. Let

$\mathcal{F} \in \omega(\mathcal{Z}_2)$. Suppose $A \in \mathcal{Z}_1 \subseteq \mathcal{Z}_2$ with $A \cap Z \neq \emptyset$ for every $Z \in \mathcal{F} \cap \mathcal{Z}_1$. By R9.1.4 and R9.1.3 it is sufficient to show $A \in \mathcal{F}$. Deny that, i.e., suppose $\mathcal{F} \notin A^{\omega_2}$. Because ϕ is one-to-one, $\phi(\mathcal{F})$ is not in the closed set $\phi[A^{\omega_2}]$. By P3.6 $\mathcal{Z}_1^{\omega_1}$ is a normal basis for $\omega(\mathcal{Z}_1)$. Thus by P1.1iii there is $B \in \mathcal{Z}_1$ such that $\phi(\mathcal{F}) \in B^{\omega_1}$ and $B^{\omega_1} \cap \phi[A^{\omega_2}] = \emptyset$. Note that $\phi[B^{\omega_2}] \subseteq B^{\omega_1}$ as follows: Let $\mathcal{G} \in B^{\omega_2}$, i.e., $B \in \mathcal{G}$. By R9.Add.3 $\mathcal{G} \cap \mathcal{Z}_1 \subseteq \phi(\mathcal{G})$ so that $B \in \phi(G)$, i.e., $\phi(\mathcal{G}) \in B^{\omega_1}$. Thus $\phi[B^{\omega_2}] \cap \phi[A^{\omega_2}] = \emptyset$. Because ϕ is one-to-one, this says $\phi[B^{\omega_2} \cap A^{\omega_2}] = \emptyset$ so that $B^{\omega_2} \cap A^{\omega_2} = (B \cap A)^{\omega_2} = \emptyset$. Thus $B \cap A = \emptyset$. But this contradicts that $\phi(\mathcal{F})$ is a $\mathcal{Z}_1$-filter, since both A and B are in $\phi(\mathcal{F})$.

Next an unexpected example (not the one desired) is given. It uses the example from Alò and Shapiro [1; p. 53] referenced after R9.1.2. A detailed presentation of that example follows, because the details are used afterward.

For the rest of this added subsection assume $\mathbb{R}$ has the usual topology and let $\mathcal{Z}^*$ be the collection of closed sets Z with the property that, if $0 \in Z$, then either 0 is not a limit point of Z or 0 is not in the boundary of Z.

The routine proof of the first lemma is left to the reader in [1].

Lemma R9.Add.7 $\mathcal{Z}^*$ is a normal basis for $\mathbb{R}$.

Proof: The four requirements in definition P3.1 will be verified. First, $\mathcal{Z}^*$ is a base for the closed sets: Let F be closed with $x \notin F$. If $x = 0$, $F \in \mathcal{Z}^*$. If $x \neq 0$, there is $\delta > 0$ with $(x - \delta, x + \delta) \subseteq \mathbb{R} - F$ and $0 \notin \{x - \delta, x + \delta\}$. Then $F \subseteq (-\infty, x - \delta] \cup [x + \delta, \infty)$, which is in $\mathcal{Z}^*$. Next let $Z_1, Z_2 \in \mathcal{Z}^*$ and assume $0 \in Z_1 \cup Z_2$. If 0 is in the interior of either Z_1 or Z_2, it is in the interior of the union and so $Z_1 \cup Z_2 \in \mathcal{Z}^*$. Now assume 0 is not in the interior of either. If $0 \in Z_1 \cap Z_2$, it is not a limit point of either and so there is $\epsilon > 0$ such that $Z_i \cap (-\epsilon, \epsilon) = \{0\}$ for $i = 1, 2$. Then $(-\epsilon, \epsilon) \cap (Z_1 \cup Z_2) = \{0\}$, i.e., $Z_1 \cup Z_2 \in \mathcal{Z}^*$. If 0 is in Z_1 but not Z_2, there is $\epsilon > 0$ such that $Z_1 \cap (-\epsilon, \epsilon) = \{0\}$ and $Z_2 \cap (-\epsilon, \epsilon) = \emptyset$. Then $(-\epsilon, \epsilon) \cap (Z_1 \cup Z_2) = \{0\}$ so that $Z_1 \cup Z_2 \in \mathcal{Z}^*$. The final case, 0 is in Z_2 but not Z_1, is similar. Thus $\mathcal{Z}^*$ is closed under finite unions. Next suppose $0 \in Z_1 \cap Z_2$. If 0 is in the interior of both, it is in the interior of the intersection so that $Z_1 \cap Z_2 \in \mathcal{Z}^*$. If 0 is not a limit point of Z_1, there is G open with $G \cap Z_1 = \{0\}$ so that $(Z_1 \cap Z_2) \cap G = \{0\}$ and again $Z_1 \cap Z_2 \in \mathcal{Z}^*$. The remaining case, 0 is not a limit point of Z_2, is similar. Thus $\mathcal{Z}^*$ is closed under finite intersections. To verify P3.1iii, for F closed and $x \notin F$, there is $\delta > 0$ with $(x - \delta, x + \delta) \subseteq \mathbb{R} - F$ and $0 \notin \{x - \delta, x + \delta\}$. Then $[x - \delta, x + \delta] \in \mathcal{Z}^*$ and $[x - \delta, x + \delta] \cap F = \emptyset$. Finally, let $Z_1, Z_2 \in \mathcal{Z}^*$ with $Z_1 \cap Z_2 = \emptyset$. Because $\mathbb{R}$ is normal, there are O_1, O_2 open with $O_1 \cap O_2 = \emptyset$ and $Z_i \subseteq O_i$ for $i = 1, 2$. If $0 \notin O_1 \cup O_2$, there is $\epsilon > 0$ such that $[-\epsilon, \epsilon] \cap Z_i = \emptyset$ for $i = 1, 2$. Let $C = (\mathbb{R} - O_1) \cup [-\epsilon, \epsilon]$ and $D = (\mathbb{R} - O_2) \cup [-\epsilon, \epsilon]$. Then C, D are in $\mathcal{Z}^*$, $C \cup D = \mathbb{R}$, $Z_1 \subseteq \mathbb{R} - C$, and $Z_2 \subseteq \mathbb{R} - D$, i.e., P3.1iv holds in this case. If $0 \in O_1$, there is $\epsilon > 0$ such that $[-\epsilon, \epsilon] \subseteq O_1$. Now let $C = \mathbb{R} - O_1$ and $D = (\mathbb{R} - O_2) \cup [-\epsilon, \epsilon]$. Again, C, D are in $\mathcal{Z}^*$, $C \cup D = \mathbb{R}$, $Z_1 \subseteq \mathbb{R} - C$, and $Z_2 \subseteq \mathbb{R} - D$. The case with $0 \in O_2$ proceeds in the same way.

Lemma R9.Add.8 Let $\mathcal{F}^*$ be the set of $Z \in \mathcal{Z}^*$ such that 0 is in the interior of Z. Then $\mathcal{F}^*$ is a prime $\mathcal{Z}^*$-filter but not a $\mathcal{Z}^*$-ultrafilter. Moreover, the unique $\mathcal{Z}^*$-ultrafilter containing $\mathcal{F}^*$ is the $\mathcal{Z}^*$-point filter of 0.

Proof: Clearly $\mathcal{F}^*$ is a non-empty collection of non-empty $\mathcal{Z}^*$-sets and it has the required superset property. For F_1, F_2 in $\mathcal{F}^*$, because the interior of an intersection is the intersection of the interiors, $F_1 \cap F_2 \in \mathcal{F}^*$. Thus $\mathcal{F}^*$ is a $\mathcal{Z}^*$-filter. Now let Z_1, Z_2 be in $\mathcal{Z}^*$

with $Z_1 \cup Z_2 \in \mathcal{F}^*$. By definition of $\mathcal{F}^*$ there is $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq Z_1 \cup Z_2$. One of the two, say Z_1, contains 0. If 0 is not in the boundary of Z_1, 0 is in the interior of Z_1 so that $Z_1 \in \mathcal{F}^*$. If 0 is in the boundary of Z_1, by definition of $\mathcal{Z}^*$, 0 is not a limit point of Z_1 so that there is $0 < \delta < \epsilon$ with $(-\delta, \delta) \cap Z_1 = \{0\}$. Then $(-\delta, 0) \cup (0, \delta) \subseteq Z_2$. Since Z_2 is closed, $0 \in Z_2$ as well so that $Z_2 \in \mathcal{F}^*$. Thus $\mathcal{F}^*$ is prime. By definition the $\mathcal{Z}^*$-point filter of 0, $\mathcal{F}_0^*$, is the set of all elements of $\mathcal{Z}^*$ containing 0. $\mathcal{F}_0^*$ is a $\mathcal{Z}^*$-ultrafilter and clearly $\mathcal{F}^* \subseteq \mathcal{F}_0^*$. $\{0\}$ is in $\mathcal{F}_0^*$ but not $\mathcal{F}^*$ and so $\mathcal{F}^*$ is not a $\mathcal{Z}^*$-ultrafilter. Uniqueness follows from R9.1.3.

Lemma R9.Add.9 Let $\mathcal{Z}$ be the collection of all closed subsets of $\mathbb{R}$. Then $\mathcal{Z}$ is a normal basis for $\mathbb{R}$ and $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is equivalent to $(\omega(\mathcal{Z}^*), \iota_{\mathcal{Z}^*})$.

Proof: In any T_4 space the collection of all closed subsets is a normal basis. Let $\mathcal{U}, \mathcal{U}^*$ be the separated totally bounded uniformities corresponding to the compactification classes of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ and $(\omega(\mathcal{Z}^*), \iota_{\mathcal{Z}^*})$ respectively. Since $\mathcal{Z}^* \subseteq \mathcal{Z}$, by R14.Add.2 and R1.5, $\mathcal{U}^* \subseteq \mathcal{U}$. Now let $U \in \mathcal{U}$. By R14.1.3 there are $Z_1, \ldots, Z_n$ in $\mathcal{Z}$ with $\cap_{i=1}^n Z_i = \emptyset$ and $\cup_{i=1}^n (\mathbb{R} - Z_i) \times (\mathbb{R} - Z_i) \subseteq U$. Let $S = \{i : 0 \notin Z_i\}$. S is non-empty because the intersection is empty. Let $T = \{i : 0 \in Z_i\}$. Since $\cup_{i \in S} Z_i$ is closed, there is $\epsilon > 0$ with $[-\epsilon, \epsilon] \subseteq \mathbb{R} - \cup_{i \in S} Z_i$. Let $Z_i^* = Z_i$ if $i \in S$ and $Z_i^* = Z_i \cup [-\epsilon, \epsilon]$ if $i \in T$. Clearly each Z_i^* is in $\mathcal{Z}^*$ and $Z_i^* \cap [-\epsilon, \epsilon] = \emptyset$ for each $i \in S$.

$$
\begin{aligned}
\cap_{i=1}^n Z_i^* &= (\cap_{i \in S} Z_i) \cap (\cap_{i \in T} (Z_i \cup [-\epsilon, \epsilon])) \\
&= (\cap_{i \in S} Z_i) \cap ((\cap_{i \in T} Z_i) \cup [-\epsilon, \epsilon])) \\
&= (\cap_{i=1}^n Z_i) \cup ((\cap_{i \in S} Z_i) \cap [-\epsilon, \epsilon]) = \emptyset.
\end{aligned}
$$

By R14.1.3 $V = \cup_{i=1}^n (\mathbb{R} - Z_i^*) \times (\mathbb{R} - Z_i*)$ is in $\mathcal{U}^*$. Since $\mathbb{R} - Z_i^* \subseteq \mathbb{R} - Z_i$ for all i, $V \subseteq U$ and so $U \in \mathcal{U}^*$. By R1.5, $\mathcal{U}^* = \mathcal{U}$ implies the conclusion.

Comment: $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is in the class of the Stone-Čech compactification. That fact does not play a role here.

The following corollary shows that the prime $\mathcal{Z}^*$-filter $\mathcal{F}^*$ cannot be obtained from a $\mathcal{Z}$-ultrafilter.

Corollary R9.Add.10 Let $\mathcal{Z}$ be the collection of all closed subsets of $\mathbb{R}$. There is no $\mathcal{F}$ in $\omega(\mathcal{Z})$ such that $\mathcal{F} \cap \mathcal{Z}^* = \mathcal{F}^*$.

Proof: This follows from R9.Add.9, R9.Add.6, and R9.Add.8.

Additional Reference

12. This website, R14: Uniformities and Normal Bases