Uniform Continuity and Extension of Maps

For a $T_{3\frac{1}{2}}$ space (X, τ), the set of totally bounded uniformities for X that generate τ will be denoted $\mathcal{TB}(X)$, as in [4]. This section examines connections between the extendability of a continuous map relative to a given compactification and its uniform continuity relative to the element of $\mathcal{TB}(X)$ associated with the compactification. Certain classes of bounded, real-valued continuous functions on X are shown to generate elements of $\mathcal{TB}(X)$.

Main Result

For the first two results in this subsection, the following will be assumed: Let (Y, f) be a T_2 compactification of the $T_{3\frac{1}{2}}$ space (X, τ). With Ψ_0 defined as in [4], let $\mathcal{U}$ be the element of $\mathcal{TB}(X)$ with $\Psi_0(\mathcal{U}) = [(Y, f)]$. Let Z be a compact T_2 space with $\mathcal{W}$ the unique uniformity which generates the topology of Z.

The following result, which characterizes continuous maps that are extendible to Y, is probably known, but I have no reference.

Theorem R7.1.1 Let $h : X \to Z$ be continuous. Then there exists a unique continuous map $H : Y \to Z$ with $H \circ f = h$ if and only if $h : (X,\mathcal{U}) \to (Z, \mathcal{W})$ is uniformly continuous.

Proof: By R1.6a, $f : (X,\mathcal{U}) \to f[X]$ is a unimorphism, where $f[X]$ has the subspace uniformity from Y. If H exists, since Y is compact, H must be uniformly continuous. Thus $h = H \circ f$ is the composition of two uniformly continuous maps and so uniformly continuous. For the converse, assume h is uniformly continuous. Then $h \circ f^{-1}$ is a uniformly continuous map from $f[X]$, a dense subset of Y, to the complete and separated space $(Z, \mathcal{W})$. By R1.1 there exists a unique uniformly continuous map $H : Y \to Z$ which extends $h \circ f^{-1}$, i.e. $H \circ f = h$ as required.

As in definition R3.3.1, the map H in the above result will be described as the continuous extension of h to Y.

Corollary R7.1.2 Let $\mathcal{U}_M$ be the largest element of $\mathcal{TB}(X)$. Then every continuous map from X to Z is uniformly continuous from $(X,\mathcal{U}_M)$ to $(Z, \mathcal{W})$.

Proof: By R1.8 in [4] $\Psi_0(\mathcal{U}_M) = [(\beta X, \iota)]$ and every continuous map from X to Z extends to βX.

Corollary R7.1.3 Let $(X_1,\mathcal{U}_1)$ and $(X_2,\mathcal{U}_2)$ be separated, totally bounded uniform spaces with $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$. Let $g : (X_1,\mathcal{U}_1) \to (X_2,\mathcal{U}_2)$ be uniformly continuous. Then there is a unique continuous map $G : Y_1 \to Y_2$ such that $G \circ f_1 = f_2 \circ g$.

Proof: As noted above, the map f_2 is uniformly continuous and so $f_2 \circ g$ is a uniformly continuous map into the compact, T_2 space Y_2. G exists by the theorem.

The last corollary can be used to introduce a covariant functor from the category of separated, totally bounded uniform spaces with uniformly continuous maps to the category of compact Hausdorff spaces with continuous maps by associating $(X,\mathcal{U})$ with a representative of $\Psi_0(\mathcal{U})$ and a uniformly continuous map with the extension guaranteed by R7.1.3.

These results also provide an alternate proof of R3.3.5 in [5], as follows.

Lemma R7.1.4 Let $(X_1,\mathcal{U}_1)$ and $(X_2,\mathcal{U}_2)$ be separated, totally bounded uniform spaces with $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$. Let $g : X_1 \to X_2$. If g has a continuous extension from Y_1

to Y_2, i.e. there is a unique continuous map $G : Y_1 \to Y_2$ such that $G \circ f_1 = f_2 \circ g$, then $g : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ is uniformly continuous.

Proof: By R1.6a, $f_1 : (X_1, \mathcal{U}_1) \to f_1[X_1]$ and $f_2 : (X_2, \mathcal{U}_2) \to f_2[X_2]$ are unimorphisms, where $f_i[X_i]$ has the subspace uniformity from the unique uniformity for Y_i. By compactness, G is uniformly continuous and, because the image of $G \circ f_1$ is contained in $f_2[X_2]$, the extension equation can be written $g = f_2^{-1} \circ (G \circ f_1)$. Thus g is a composition of uniformly continuous maps and so uniformly continuous.

Proposition R7.1.5 Let (X, τ) and (W, σ) be a $T_{3\frac{1}{2}}$ spaces. Let Δ be a non-empty set. Let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ and $\{(Z_\alpha, g_\alpha) : \alpha \in \Delta\}$ be collections of T_2 compactifications of X and W respectively. Let $h : X \to W$ be continuous and assume, for every $\alpha \in \Delta$, h has an extension $h_\alpha : Y_\alpha \to Z_\alpha$. Then there exists $H : \vee Y_\alpha \to \vee Z_\alpha$, which is an extension of h.

Proof: For each $\alpha \in \Delta$ let $\mathcal{U}_\alpha$ in $\mathcal{TB}(X)$ and $\mathcal{V}_\alpha$ in $\mathcal{TB}(W)$ be the uniformities such that $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, f_\alpha)]$ and $\Psi_0(\mathcal{V}_\alpha) = [(Z_\alpha, g_\alpha)]$. For each $\alpha \in \Delta$ by R7.1.4 $h : (X, \mathcal{U}_\alpha) \to (W, \mathcal{V}_\alpha)$ is uniformly continuous. Since $(h \times h)^{-1}(\cap_{i=1}^n V_i) = \cap_{i=1}^n (h \times h)^{-1}[V_i]$ where $V_i \in \cup\{\mathcal{V}_\alpha : \alpha \in \Delta\}$, it follows easily that $h : (X, \vee \mathcal{U}_\alpha) \to (W, \vee \mathcal{V}_\alpha)$ is uniformly continuous. By R1.5 $\Psi_0(\vee \mathcal{U}_\alpha)$ is the compactification class of $\vee Y_\alpha$ and $\Psi_0(\vee \mathcal{V}_\alpha)$ is the compactification class of $\vee Z_\alpha$. By R7.1.3 the required extension H exists.

Generating $\mathcal{U}$ from Families of Maps

For this subsection some background material on uniform spaces not included in [3] is needed. The first few results describe weak uniformities induced by a map and the relation to the corresponding weak topologies. The straightforward proofs can be found in [2].

Definition R7.2.1 Let $(Y, \mathcal{W})$ be a uniform space and let $f : X \to Y$. $f^{-1}\mathcal{W}$ is defined to be $\{U \subseteq X \times X : \text{there is } W \in \mathcal{W} \text{ with } (f \times f)^{-1}[W] \subseteq U\}$.

Proposition R7.2.2 Let $(Y, \mathcal{W})$ be a uniform space and let $f : X \to Y$. Then $f^{-1}\mathcal{W}$ is the smallest uniformity for X which makes f uniformly continuous.

$f^{-1}\mathcal{W}$ is sometimes called the weak uniformity induced by f and $\mathcal{W}$ or, if there is no ambiguity about the uniformity on Y, simply the weak uniformity induced by f.

Definition R7.2.3 Let (Y, σ) be a topological space and let $f : X \to Y$. $f^{-1}\sigma$ is defined to be $\{f^{-1}[O] : O \in \sigma\}$.

Proposition R7.2.4 Let (Y, σ) be a topological space and let $f : X \to Y$. $f^{-1}\sigma$ is the smallest topology for X which makes f continuous.

As in the uniform space case, $f^{-1}\sigma$ is sometimes called the weak topology induced by f and σ or simply the weak topology induced by f.

Proposition R7.2.5 Let $(Y, \mathcal{W})$ be a uniform space and let $f : X \to Y$. Then $\tau(f^{-1}\mathcal{W}) = f^{-1}\tau(\mathcal{W})$.

Proposition R7.2.6 Let $(Y, \mathcal{W})$ be a uniform space and let $f : X \to Y$. If $f[X]$ with the subspace uniformity from $\mathcal{W}$ is totally bounded, then $(X, f^{-1}\mathcal{W})$ is also totally bounded.

Proof: If $W \in \mathcal{W}$, then $V = W \cap (f[X] \times f[X])$ is the resulting element of the subspace uniformity. Since $f[X]$ is totally bounded, there exist $x_1, \ldots, x_n$ in X such that $f[X] = \cup_{i=1}^n V[f(x_i)]$. It follows easily that $X = \cup_{i=1}^n (f \times f)^{-1}[W][x_i]$, which yields the desired conclusion.

For the rest of this subsection $\mathcal{V}$ will denote the canonical uniformity on $\mathbb{R}$, i.e., the uniformity generated by the absolute value metric. The facts above combined with P2.13 and P2.14 in [3] suggest using a family of continuous, bounded, real-valued functions to generate a totally bounded uniformity and consequently, if certain separation properties are satisfied, a compactification. In Chapter 2 of [1] Chandler uses suitable families to construct the compactifications directly, i.e. without uniformities as an intermediate step.

Definition R7.2.7 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{C}$ be a non-empty set of continuous, bounded real-valued maps from X. $\mathcal{U}(\mathcal{C}) = \bigvee\{f^{-1}\mathcal{V} : f \in \mathcal{C}\}$.

Proposition R7.2.8 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{C}$ be a non-empty set of continuous, bounded real-valued maps from X. Then $\mathcal{U}(\mathcal{C})$ is a totally bounded uniformity for X and $\tau(\mathcal{U}(\mathcal{C})) \subseteq \tau$. In addition each $f \in \mathcal{C}$ is uniformly continuous from $(X,\mathcal{U}(\mathcal{C}))$.

Proof: For each $f \in \mathcal{C}$, $f[X]$ is bounded in $\mathbb{R}$ and so also totally bounded. By R7.2.6 each $f^{-1}\mathcal{V}$ is totally bounded and so by P2.13 $\mathcal{U}(\mathcal{C})$ is also totally bounded. Since each f is continuous, the weak topology induced by f, which is $\tau(f^{-1}\mathcal{V})$ by R7.2.5, must be contained in τ. By P2.14 $\tau(\mathcal{U}(\mathcal{C})) \subseteq \tau$. Finally each f is uniformly continuous from $(X, f^{-1}\mathcal{V})$ by R7.2.2 and so also uniformly continuous with any larger uniformity on X.

A family $\mathcal{C}$ of continuous real-valued functions on X is said to separate points and closed sets if, given F closed in X and $x \in X - F$, there is $f \in \mathcal{C}$ such that $\{f(x)\} \cap \overline{f[F]} = \emptyset$, where $\overline{f[F]}$ denotes the closure of $f[F]$ in $\mathbb{R}$.

Proposition R7.2.9 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{C}$ be a non-empty set of continuous, bounded real-valued maps from X. If $\mathcal{C}$ separates points and closed sets, then $\tau(\mathcal{U}(\mathcal{C})) = \tau$.

Proof: Given $x \in O \in \tau$, the hypothesis implies there is $f \in \mathcal{C}$ and G open in $\mathbb{R}$ with $x \in f^{-1}[G] \subseteq O$. Since $f^{-1}[G] \in \tau(\mathcal{U}(\mathcal{C}))$, the conclusion follows.

Proposition R7.2.10 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{C}$ be a non-empty set of continuous, bounded real-valued maps from X. Assume $\mathcal{C}$ separates points and closed sets. Then $\mathcal{U}(\mathcal{C}) \in \mathcal{TB}(X)$. In addition, if $[(Y, g)] = \Psi_0(\mathcal{U}(\mathcal{C}))$, then every f in $\mathcal{C}$ has a continuous extension to Y.

Proof: The first conclusion summarizes R7.2.9 and part of R7.2.8. The second conclusion is immediate from R7.2.8 and R7.1.1.

Examples R7.2.11 Let $(X,\mathcal{U})$ be $[0, 1)$ with the subspace uniformity from $(\mathbb{R}\ , \mathcal{V})$. It can be shown that $\mathcal{U}$ is the smallest element of $\mathcal{TB}(X)$, i.e. the uniformity $\mathcal{U}_m$ associated with the one point compactification. Let $I : X \to \mathbb{R}$ by $I(x) = x$. $\{I\}$ separates points and closed sets as defined above and so $\mathcal{U}(\{I\}) \in \mathcal{TB}(X)$. Thus $\mathcal{U}_m \subseteq \mathcal{U}(\{I\})$, and the reverse containment holds since I is uniformly continuous from $(X,\mathcal{U}_m)$. In this example the set of uniformly continuous maps from $(X,\mathcal{U}(\{I\}))$ is strictly larger than the generating family $\{I\}$. As a second example, let $\mathcal{C} = \{f, g\}$, where $f(x) = 2x$ for $0 \le x \le \frac{1}{2}$, $f(x) = 1$ for $\frac{1}{2} < x < 1$, $g(x) = 1$ for $0 \le x \le \frac{1}{2}$, and $g(x) = 2 - 2x$ for $\frac{1}{2} < x < 1$. It can be shown that $f^{-1}\mathcal{V} \vee g^{-1}\mathcal{V}$ is $\mathcal{U}_m$, but $\mathcal{C}$ can not separate $\frac{1}{2}$ and $[0, \frac{1}{3}] \cup [\frac{2}{3}, \frac{3}{4}]$. Thus the conclusions of R7.2.10 may hold even if $\mathcal{C}$ does not separate points and closed sets.

Definition R7.2.12 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U} \in \mathcal{TB}(X)$. $\mathcal{C}(\mathcal{U})$ is defined to be $\{f : (X,\mathcal{U}) \to (\mathbb{R}\ , \mathcal{V}) : f$ is uniformly continuous$\}$.

Since $(X,\mathcal{U})$ is totally bounded, each element of $\mathcal{C}(\mathcal{U})$ is a bounded continuous map. The following well-known lemmas will lead to the conclusion that $\mathcal{C}(\mathcal{U})$ separates points and closed sets. Notation intoduced in [3] for a uniformity generated by a pseudometric will be used. The fact that a uniformity is pseudo-metrizable if and only if it has a countable base (Theorem 11.5.1 in [2]) is also needed.

Lemma R7.2.13 Let $(X,\mathcal{U})$ be a uniform space and let $U \in \mathcal{U}$. Then there exists a pseudo-metric d for X such that $U \in \mathcal{U}_d \subseteq \mathcal{U}$.

Proof: By the definition of a uniformity a sequence of entourages $\{V_n\}$ can be found in $\mathcal{U}$ with $V_n = V_n^{-1}$, $V_n \circ V_n \subseteq V_{n-1}$ for $n \geq 2$, and $V_1 \circ V_1 \subseteq U$. It is easy to check that $\mathcal{W} = \{W \subseteq X \times X : V_n \subseteq W$ for some $n\}$ is a uniformity for X. By the theorem just mentioned, there is a pseudo-metric for X with $\mathcal{U}_d = \mathcal{W}$. Clearly $U \in \mathcal{W} \subseteq \mathcal{U}$.

Lemma R7.2.14 Let d be a pseudo-metric for X and let S be a non-empty subset of X. Let $f : X \rightarrow \mathbb{R}$ be defined by $f(x) = \inf\{d(x,t) : t \in S\}$.
Then $f : (X,\mathcal{U}_d) \rightarrow (\mathbb{R}\ ,\mathcal{V})$ is uniformly continuous.

Proof: $|f(a) - f(b)| \leq d(a,b)$ for any $a, b \in X$.

Proposition R7.2.15 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U} \in \mathcal{TB}(X)$. Then $\mathcal{C}(\mathcal{U})$ separates points and closed sets.

Proof: Let F be closed relative to τ and let $x \in X - F$. There is $U \in \mathcal{U}$ with $U[x] \subseteq X - F$. By R7.2.13 there is a pseudo-metric d for X such that $U \in \mathcal{U}_d \subseteq \mathcal{U}$. There is $\epsilon > 0$ with $V_\epsilon \subseteq U$. For $\delta = \epsilon/2$, let $f : X \rightarrow \mathbb{R}$ by $f(t) = \inf\{d(t,y) : y \in V_\delta[x]\}$. By R7.2.14 f is uniformly continuous from $(X,\mathcal{U}_d)$ and so also from $(X,\mathcal{U})$. Clearly $f(x) = 0$ and $f(t) \geq \delta > 0$ for all $t \in F$ so that $\{f(x)\} \cap \overline{f[F]} = \emptyset$.

Proposition R7.2.16 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U} \in \mathcal{TB}(X)$.
Then $\mathcal{U}(\mathcal{C}(\mathcal{U})) = \mathcal{U}$.

Proof: Since each $f \in \mathcal{C}(\mathcal{U})$ is uniformly continuous from $(X,\mathcal{U})$, $f^{-1}\mathcal{V} \subseteq \mathcal{U}$ and so $\mathcal{U}(\mathcal{C}(\mathcal{U})) \subseteq \mathcal{U}$. For the converse, let $U \in \mathcal{U}$ and let d be a pseudo-metric with $U \in \mathcal{U}_d \subseteq \mathcal{U}$. There is $\epsilon > 0$ with the ϵ-strip $V_\epsilon(d) = \{(x,y) : d(x,y) < \epsilon\} \subseteq U$. Let $\delta = \epsilon/3$. Since $\mathcal{U}$ is totally bounded, there exist $x_1, \ldots, x_n$ with $\cup_{i=1}^n V_\delta(d)[x_i] = X$. For $i = 1, \ldots, n$ let $f_i(x) = d(x, x_i)$. Each f_i is uniformly continuous and so $\cap_{i=1}^n (f_i \times f_i)^{-1}[V_\delta]$ is in $\mathcal{U}(\mathcal{C}(\mathcal{U}))$, where V_δ is the δ-strip determined by the absolute value metric on $\mathbb{R}$. Let $(a,b) \in \cap_{i=1}^n (f_i \times f_i)^{-1}[V_\delta]$. It is sufficient to show $(a,b) \in U$. There is j such that $a \in V_\delta(d)[x_j]$. Since $(a,b) \in (f_j \times f_j)^{-1}[V_\delta]$, $d(b,x_j) < d(a,x_j) + \delta$. Thus $d(a,b) < 3\delta = \epsilon$ and so $(a,b) \in U$.

Proposition R7.2.17 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U} \in \mathcal{TB}(X)$. Then $\mathcal{C}(\mathcal{U}) = \cup\{\mathcal{C} : \mathcal{U}(\mathcal{C}) = \mathcal{U}\}$, i.e., $\mathcal{C}(\mathcal{U})$ is the largest collection of continuous, bounded, real-valued functions generating $\mathcal{U}$.

Proof: For any such collection $\mathcal{C}$, every map in $\mathcal{C}$ is uniformly continuous from $(X,\mathcal{U})$ by R7.2.8 so that $\mathcal{C} \subseteq \mathcal{C}(\mathcal{U})$. By R7.2.16 $\mathcal{C}(\mathcal{U})$ is one such collection.

Proposition R7.2.18 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U}_1$ and $\mathcal{U}_2$ be in $\mathcal{TB}(X)$. Then $\mathcal{U}_1 \subseteq \mathcal{U}_2$ if and only if $\mathcal{C}(\mathcal{U}_1) \subseteq \mathcal{C}(\mathcal{U}_2)$.

Proof: Assume $\mathcal{C}(\mathcal{U}_1) \subseteq \mathcal{C}(\mathcal{U}_2)$. Clearly from the definition $\mathcal{U}(\mathcal{C}(\mathcal{U}_1)) \subseteq \mathcal{U}(\mathcal{C}(\mathcal{U}_2))$, i.e., by R7.2.16, $\mathcal{U}_1 \subseteq \mathcal{U}_2$. The converse is immediate from the definition of uniform continuity.

The collection $\mathcal{C}(\mathcal{U})$ must contain constant maps, separate points and closed sets, be a vector space under pointwise operations, and be a closed set relative to the topology of uniform convergence determined by $(\mathbb{R}\ ,\mathcal{V})$. The following example shows that those properties are not sufficient to identify a collection as $\mathcal{C}(\mathcal{U})$ for some $\mathcal{U}$.

Example R7.2.19 Let $X = [0,1)$ with uniformity $\mathcal{U}_m$ as in R7.2.11, and let $\mathcal{C} =$

$\{l : X \to \mathbb{R} : l(x) = ax + b$ for some a, b in $\mathbb{R}$ $\}$. Since the map I as in R7.2.11 is in $\mathcal{C}$, the collection separates points and closed sets. It clearly contains all constant maps and is a vector space. It is easy to check that $\mathcal{C}$ is closed relative to the topology of uniform convergence. Also $\mathcal{C}$ is a proper subset of $\mathcal{C}(\mathcal{U}_m)$ so that $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{U}_m$. By R7.2.10 $\mathcal{U}(\mathcal{C}) \in \mathcal{TB}(X)$ and, since $\mathcal{U}_m$ is the smallest element of $\mathcal{TB}(X)$, $\mathcal{U}_m \subseteq \mathcal{U}(\mathcal{C})$.

Albert J. Klein 2004
http://www.susanjkleinart.com/compactification/

References

1. Chandler, R. E., Hausdorff Compactifications, Marcel Dekker Inc., 1976
2. Wilansky, A., Topology for Analysis ,Ginn and Co., 1970.
3. This website, P2: Uniform Spaces
4. This website, R1: Existence of the Supremum via Uniform Space Theory
5. This website, R3: Representation of Suprema

Added 2021

This added subsection starts by recording two easy corollaries of previously obtained results. The first answers the question of whether, given a uniform space and a compactification of the underlying $T_{3\frac{1}{2}}$ space, the embedding is uniformly continuous.

Proposition R7.Add.1 Let (Y, f) be a T_2 compactification of the $T_{3\frac{1}{2}}$ topological space (X, τ), and let $\mathcal{U}$ be the separated, totally bounded uniformity for X corresponding to the compactification class $[(Y, f)]$. Let $\mathcal{V}$ be a uniformity on X, not necessarily totally bounded, such that $\tau(\mathcal{V}) = \tau$. Then $f : (X, \mathcal{V}) \to Y$ is uniformly continuous if and only if $\mathcal{U} \subseteq \mathcal{V}$.

Proof: Let $\mathcal{W}$ be the unique uniformity for the compact, T_2 space Y. By R1.6a f is a uniform embedding from $(X,\mathcal{U})$ to $(Y, \mathcal{W})$ and so $\mathcal{U} = f^{-1}\mathcal{W}$. Since $f : (X, \mathcal{V}) \to (Y, \mathcal{W})$ is uniformly continuous if and only if $f^{-1}\mathcal{W} \subseteq \mathcal{V}$, the conclusion follows.

The second deals with extendibility of bounded continuous real-valued functions to a compactification, where extendibility means factoring through the embedding as in R7.1.1. Here $\mathbb{R}$ is assumed to have the usual topology and boundedness of a map means its image is a bounded set relative to the usual metric on $\mathbb{R}$.

Proposition R7.Add.2 Let (Y, f) be a T_2 compactification of the $T_{3\frac{1}{2}}$ topological space (X, τ), and let $\mathcal{U}$ be the separated, totally bounded uniformity for X corresponding to the compactification class $[(Y, f)]$. Let $g : (X, \tau) \to \mathbb{R}$ be a bounded continuous map. Let $\mathcal{V}$ be a uniformity for $\mathbb{R}$, not necessarily totally bounded, which generates the usual topology. Then g is continuously extendible to Y if and only if $g : (X,\mathcal{U}) \to (\mathbb{R}, \mathcal{V})$ is uniformly continuous.

Proof: Let a, b be real numbers such that $g[X] \subseteq [a, b]$. Then g is continuous into $[a, b]$ and so by R7.1.1 g is continuously extendible to Y if and only if g is uniformly continuous from $(X,\mathcal{U})$ to $[a, b]$ with its unique uniformity. Also $g : (X,\mathcal{U}) \to (\mathbb{R}, \mathcal{V})$ is uniformly continuous if and only if g is uniformly continuous from $(X,\mathcal{U})$ to $[a, b]$ with the subspace uniformity from $\mathcal{V}$. Since the subspace uniformity from $\mathcal{V}$ generates the usual topology on $[a,b]$, it must be the unique uniformity on $[a,b]$. The conclusion follows.

Next the question of extensions willl be considered from a slightly different point of view. Now a separated uniform space, not necessarily totally bounded, is given. The goal is to find a compactification for which every uniformly continuous function extends. Of couse, the Stone-Čech compactification is the largest such. The following identifies one that could be regarded as the compactification determined by the given uniformity.

This requires some facts presented in the added subsection of [6]. The arguments there depend only on the basic facts of [3]. The two items needed are :

Definition R8.Add.3 Let $(X,\mathcal{U})$ be a uniform space. A proximal cover of X is a finite collection $\{A_1,\ldots,A_n\}$ of subsets of X for which there exist sets $B_1,\ldots,B_n$ and $U \in \mathcal{U}$ such that $X = \cup_{i=1}^n B_i$ and $U[B_i] \subseteq A_i$ for $1 \le i \le n$.

Lemma R8.Add.5 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{V}$ be defined as the set $\{V \subseteq X \times X$: there is a proximal cover $\{A_1,\ldots,A_n\}$ with $\cup_{i=1}^n A_i \times A_i \subseteq V\}$. Then

i) $\mathcal{V}$ is a uniformity for X.
ii) $\mathcal{V}$ is totally bounded.
iii) $\mathcal{V} \subseteq \mathcal{U}$.
iv) $\tau(\mathcal{U}) = \tau(\mathcal{V})$.

The uniformity $\mathcal{V}$ of the previous lemma will be denoted $\text{prox}(\mathcal{U})$. An example will not be given here but it is possible that $\text{prox}(\mathcal{U}) = \text{prox}(\mathcal{V})$ with $\mathcal{U} \ne \mathcal{V}$.

Corollary R7.Add.3 Let $(X,\mathcal{U})$ be a separated uniform space. Then $(X, \text{prox}(\mathcal{U}))$ is also separated.

Proof: A uniformity is separated if and only if it generates a T_2 topology. By the lemma both $\mathcal{U}$ and $\text{prox}(\mathcal{U})$ generate the same topology.

Now we note that $\text{prox}(\mathcal{U})$ is the largest totally bounded uniformity contained in $\mathcal{U}$.

Lemma R7.Add.4 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{W}$ be a totally bounded uniformity for X with $\text{prox}(\mathcal{U}) \subseteq \mathcal{W} \subseteq \mathcal{U}$. Then $\text{prox}(\mathcal{U}) = \mathcal{W}$.

Proof: Let $W \in \mathcal{W}$ and pick $W_1 = W_1^{-1}$ and $W_2 = W_2^{-1}$ in $\mathcal{W}$ with $W_1 \circ W_1 \subseteq W$ and $W_2 \circ W_2 \subseteq W_1$. Since $\mathcal{W}$ is totally bounded, there exist $x_1,\ldots,x_k$ in X such that $X = \cup_{i=1}^k W_2[x_i]$. The collection $\{W_1[x_1],\ldots,W_1[x_k]\}$ is seen to be a proximal cover of X by letting $B_i = W_2[x_i]$ and using $W_2 \circ W_2 \subseteq W_1$. Also, $\cup_{i=1}^k (W_1[x_i] \times W_1[x_i]) \subseteq W$ because $W_1 \circ W_1 \subseteq W$. By definition W is in $\text{prox}(\mathcal{U})$.

Corollary R7.Add.4 Let $(X,\mathcal{U})$ be a totally bounded uniform space. Then $\text{prox}(\mathcal{U}) = \mathcal{U}$.

Proof: Immediate from the previous lemma.

Corollary R7.Add.5 Let $(X,\mathcal{U})$ be a uniform space and let $f : X \to Y$. Let $\mathcal{V}$ be a totally bounded uniformity for Y. Then $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous if and only if $f : (X, \text{prox}(\mathcal{U})) \to (Y,\mathcal{V})$ is uniformly continuous.

Proof: Assume $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous. Then the totally bounded uniformity $f^{-1}\mathcal{V}$ is contained in $\mathcal{U}$, and so $\text{prox}(\mathcal{U}) \vee f^{-1}\mathcal{V}$ is also a totally bounded uniformity contained in $\mathcal{U}$. By the lemma $f^{-1}\mathcal{V} \subseteq \text{prox}(\mathcal{U})$, i.e., $f : (X, \text{prox}(\mathcal{U})) \to (Y,\mathcal{V})$ is uniformly continuous. The converse is clear because $\text{prox}(\mathcal{U}) \subseteq \mathcal{U}$.

In the next result the compact T_2 spaces are presumed to have their unique uniformities.

Proposition R7.Add.6 Let $(X,\mathcal{U})$ be a separated uniform space and let (Y,f) be a T_2-compactification in the compactification class corresponding to $\text{prox}(\mathcal{U})$. For any compact T_2 space K and any $g : X \to K$, $g : (X,\mathcal{U}) \to K$ is uniformly continuous if and only if g extends continuously to Y.

Proof: Let K and g be given. By R7.1.1 g extends continuously to Y if and only if $g : (X, \text{prox}(\mathcal{U})) \to K$ is uniformly continuous. The conclusion now follows from R7.Add.5.

Additional Reference

6. This website, R8: Lattice and Semilattice Properties

Added 2022

This addition records the fact that R7.1.3 can easily be strengthened to a characterization.

Proposition R7.Add.7 Let $(X_1,\mathcal{U}_1)$ and $(X_2,\mathcal{U}_2)$ be separated, totally bounded uniform spaces with $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$. Let $g : X_1 \to X_2$. Then there is a unique continuous map $G : Y_1 \to Y_2$ such that $G \circ f_1 = f_2 \circ g$ if and only if $g : (X_1,\mathcal{U}_1) \to (X_2,\mathcal{U}_2)$ is uniformly continuous.

Proof: R7.1.3 does the part when g is uniformly continuous. For the converse assume G exists. By R1.6a each f_i is a uniform embedding, i.e., unimorphism, from $(X_i,\mathcal{U}_i)$ onto $f_i[X_i]$, where $f_i[X_i]$ has the subspace uniformity from the unique uniformity for Y_i. By compactness G is uniformly continuous, as is the restriction $G|_{f_1[X_1]}$, which maps into $f_2[X_2]$. Thus $g = f_2^{-1} \circ G|_{f_1[X_1]} \circ f_1$ is a composition of uniformly continuous maps and so uniformly continuous.