Auto-homeomorphism Examples for $[0,\infty)$

For $n \in \mathbb{N}$ with $n \geq 2$, the nth root function is an auto-homeomorphism on $[0,\infty)$ with the usual topology. This section first identifies a totally bounded uniformity $\mathcal{U}$ for $[0,\infty)$ such that each nth root function is uniformly continuous but not a unimorphism. Using a construction from [4] and a supremum, a totally bounded uniformity $\mathcal{V}$ for $[0,\infty)$ is constructed such that each nth root function is a unimorphism. The compactification class corresponding to $\mathcal{V}$ is not the class of the Stone-Čech compactification.

An Example Generalized

In the first added subsection of [4] a totally bounded uniformity is constructed on $(1,\infty)$ such that the homeomorphism $x \mapsto \sqrt{x}$ is uniformly continuous but not a unimorphism. That this construction can be easily modified for any nth root on $[0,\infty)$ is shown in detail in what follows. For all n, the same totally bounded uniformity is used on $[0,\infty)$: the uniformity generated by proximal covers from the absolute value uniformity.

Given a uniform space $(X,\mathcal{U})$ and $S \subseteq X$, $\mathcal{U}_S$ will denote the subspace uniformity for S from $\mathcal{U}$. If d is a pseudo-metric on X, $\mathcal{U}_d$ denotes the uniformity on X generated by d. For $S,T \subseteq X$ with $S \neq \emptyset$ and $T \neq \emptyset$, $\text{dist}(S,T) = \inf\{d(s,t) : s \in S, t \in T\}$.

Lemma R34.1.1 Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces and let $f : X \to Y$. Assume $X = A \cup B$, $f|_A : (A,\mathcal{U}_A) \to (Y,\mathcal{V})$ is uniformly continuous, and $f|_B : (B,\mathcal{U}_B) \to (Y,\mathcal{V})$ is uniformly continuous. If there is $U_0 \in \mathcal{U}$ such that $U_0 \cap ((A-B)\times(B-A)) = \emptyset$, then $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$is uniformly continuous.

Proof: Assume U_0 exists and let $V \in \mathcal{V}$. By hypothesis, there exist $U_1, U_2 \in \mathcal{U}$ such that $U_1 \cap (A \times A) \subseteq (f|_A \times f|_A)^{-1}[V]$ and $U_2 \cap (B \times B) \subseteq (f|_B \times f|_B)^{-1}[V]$. Let $U = U_0 \cap (U_0)^{-1} \cap U_1 \cap U_2$, which is in $\mathcal{U}$. Let $(s,t) \in U$. If $\{s,t\} \subseteq A$, $(s,t) \in U_1$. By the choice of U_1, $(f(s), f(t)) \in V$. Similarly, if $\{s,t\} \subseteq B$, then $(f(s), f(t)) \in V$. Now assume $\{s,t\} \nsubseteq A$ and $\{s,t\} \nsubseteq B$. Then, because $X = A \cup B$, $(s,t) \in ((A-B)\times(B-A))$ or $(t,s) \in ((A-B)\times(B-A))$. Since (s,t) and (t,s) are in $U_0 \cap (U_0)^{-1}$, this contradicts the choice of U_0 and so the third case cannot occur. Thus $U \subseteq (f \times f)^{-1}[V]$ and so f is uniformly continuous.

Corollary R34.1.2 Let X be a set and d a pseudo-metric on X. Let $(Y,\mathcal{V})$ be a uniform space and let $f : X \to Y$. Assume $X = A \cup B$ and that $f|_A$ and $f|_B$ are uniformly continuous with the subspace uniformities from $\mathcal{U}_d$ on A and B. Suppose $A - B \neq \emptyset$ and $B - A \neq \emptyset$. If $\text{dist}(A-B, B-A) > 0$, then $f : (X,\mathcal{U}_d) \to (Y,\mathcal{V})$ is uniformly continuous.

Proof: Let $\delta = \text{dist}(A-B, B-A) > 0$. Then $V_\delta \in \mathcal{U}_d$. If (r,s) is a point in $V_\delta \cap ((A-B)\times(B-A))$, then $d(r,s) < \delta$, which contradicts the definition of the infimum. Thus $V_\delta \cap ((A-B)\times(B-A)) = \emptyset$ and the conclusion follows from the lemma.

Note that that the corollary can be derived easily with a routine ϵ, δ argument. If $A - B = \emptyset$ or $B - A = \emptyset$, uniform continuity follows because $A = X$ or $B = X$.

The following notation will be in the rest of this section. $\mathcal{U}_a$ denotes the uniformity on $[0,\infty)$ generated by the absolute value metric. For $n \in \mathbb{N}$ with $n \geq 2$, $r_n : [0,\infty) \to [0,\infty)$ by $r_n(x) = \sqrt[n]{x}$ and $p_n : [0,\infty) \to [0,\infty)$ by $p_n(x) = x^n$. From calculus r_n and p_n are continuous. Clearly, $r_n \circ p_n$ and $p_n \circ r_n$ equal the identity function on $[0,\infty)$ so that r_n is an autohomeomorphism of $[0,\infty)$ with inverse function p_n.

Lemma R34.1.3 Let $n \in \mathbb{N}$ with $n \geq 2$. Then $r_n : ([0,\infty),\mathcal{U}_a) \to ([0,\infty),\mathcal{U}_a)$ is uniformly continuous.

Proof: The continuity of r_n and the compactness of $[0,2]$ imply that $r_n|_{[0,2]}$ is uniformly continuous. For $x > 0$ the derivative $r_n'(x)$ is $\frac{1}{n\sqrt[n]{x^{n-1}}}$, which is decreasing. On $[1,\infty)$ the maximum value of r_n' is $\frac{1}{n}$. By the Mean Value Theorem, for $x, y \in [1,\infty)$ thre is m between x and y such that $r_n(x) - r_n(y) = r_n'(m)(x-y)$. Thus $|r_n(x) - r_n(y)| \leq \frac{1}{n}|x-y|$, which implies that $r_n|_{[1,\infty)}$ is uniformly continuous. R34.1.2 applies with $A = [0,2]$ and $B = [1,\infty)$. Thus $r_n : ([0,\infty),\mathcal{U}_a) \to ([0,\infty),\mathcal{U}_a)$ is uniformly continuous.

For the rest of this section let $\mathcal{U}$ denote the uniformity on $[0,\infty)$ generated by the $\mathcal{U}_a$-proximal covers. As shown in R8.Add.5, $\mathcal{U}$ is totally bounded and contained in $\mathcal{U}_a$, snd $\tau(\mathcal{U}) = \tau(\mathcal{U}_a)$.

Proposition R34.1.4 Let $n \in \mathbb{N}$ with $n \geq 2$. Then $r_n : ([0,\infty),\mathcal{U}) \to ([0,\infty),\mathcal{U})$ is uniformly continuous.

Proof: This follows from the previous lemma and R32.Add.2.

Recall that $x^n - y^n = (x-y)\Sigma_{i=0}^{n-1} x^i y^{n-1-i}$ and write $q_n(x,y) = \Sigma_{i=0}^{n-1} x^i y^{n-1-i}$.

Lemma R34.1.5 Let n be a positive integer with $n \geq 2$.
Then $\lim_{j\to\infty} q_n(\sqrt[n]{2j+1.5}, \sqrt[n]{2j+0.5}) = \infty$.

Proof: Since the coefficients of $q_n(x,y)$ are all 1, for $x, y > 0$, $q_n(x,y) \geq x^{n-1}$. If $x \geq 1$ also, $q_n(x,y) \geq x$. Combining this with the fact that $\lim_{j\to\infty} \sqrt[n]{2j+1.5} = \infty$ yields the conclusion.

Proposition R34.1.6 Let $n \in \mathbb{N}$ with $n \geq 2$. Then p_n is not uniformly continuous from $([0,\infty),\mathcal{U})$ to $([0,\infty),\mathcal{U})$.

Proof: Let $B_1 = [0,2] \cup (\cup_{i=1}^{\infty}[2i+1, 2i+2]$ and $B_2 = [0,\infty) - B_1 = \cup_{i=1}^{\infty}(2i, 2i+1)$. By definition $\{V_{0.1}[B_1], V_{0.1}[B_2]\}$ is a $\mathcal{U}_a$-proximal cover and so an element of $\mathcal{U}$ can be defined by $W = (V_{0.1}[B_1] \times V_{0.1}[B_1]) \cup (V_{0.1}[B_2] \times V_{0.1}[B_2])$. It will be shown that $(p_n \times p_n)^{-1}[W]$ is not in $\mathcal{U}$. Let $\gamma < 0$. By the previous lemma there is $N \in \mathbb{N}$ such that

$$\frac{1}{q_n(\sqrt[n]{2N+1.5}, \sqrt[n]{2N+0.5})} < \gamma.$$

Let $s = \sqrt[n]{2N+1.5}$ and $t = \sqrt[n]{2N+0.5}$. Note that s^n is in B_1 but not in $V_{0.1}[B_2]$. Also t^n is in B_2 but not in $V_{0.1}[B_1]$. Thus $(s^n, t^n) \notin W$. Since $|s^n - t^n| = |s-t|q_n(s,t)$ and $|s^n - t^n| = 1$, by the choice of N, $|s-t| = \frac{1}{q_n(s,t)} < \delta$, i.e., $(s,t) \in V_\delta$. Thus V_δ is not a subset of $(p_n \times p_n)^{-1}[W]$ for any $\delta > 0$, i.e., $(p_n \times p_n)^{-1}[W] \notin \mathcal{U}_a$. Since $\mathcal{U} \subseteq \mathcal{U}_a$, $(p_n \times p_n)^{-1}[W] \notin \mathcal{U}$. Thus the conclusion holds.

Corollary R34.1.7 Let $n \in \mathbb{N}$ with $n \geq 2$. Then the homeomorphism r_n is uniformly continuous from $([0,\infty),\mathcal{U})$ to $([0,\infty),\mathcal{U})$ but not a unimorphism.

Proof: Since p_n is the inverse function of r_n, this follows from R34.1.4 and R34.1.6.

Corollary R34.1.8 Let $n \in \mathbb{N}$ with $n \geq 2$ and let (Y, f) be a compactification of $[0,\infty)$ in the class corresponding to $\mathcal{U}$. Then the homeomorphim r_n extends continuously to Y but the extension is not a homeomorphism of Y.

Proof: By R7.1.3 and R34.1.4 r_n extends continuously. By R34.1.7 and R32.1.2 the extension is not a homeomorphism of Y.

The Uniformity $\mathcal{V}$

For the rest of this section, for $n \in \mathbb{N}$ with $n \geq 2$, $\mathcal{V}_n(\mathcal{U})$ will denote $\mathcal{V}_{r_n}(\mathcal{U})$, the separated and totally bounded uniformity on $[0, \infty)$ generated from r_n and $\mathcal{U}$ as in R32.2.14. $\mathcal{V}$ will denote $\vee\{\mathcal{V}_n(\mathcal{U}) : n \geq 2\}$.

Note that $\mathcal{U} \subseteq \mathcal{V}$ so that $\mathcal{V}$ is separable. By P2.13, P2.14, and R32.2.15 $\mathcal{V}$ is totally bounded and $\tau(\mathcal{V}) = \tau(\mathcal{U})$, the usual topology on $[0, \infty)$.

The next few results show that r_n is a unimorphism from $([0, \infty), \mathcal{V})$ to $([0, \infty), \mathcal{V})$ for all $n \in \mathbb{N}$ with $n \geq 2$

Lemma R34.2.1 Let $(A, \mathcal{W})$ be a separated, totally bounded uniform space. Let f, g be auto-homeomorphisms of $(A, \tau(\mathcal{W}))$ such that both f and g are uniformly continuous relative to $(A, \mathcal{W})$. If $f \circ g = g \circ f$, then $g : (A, \mathcal{V}_f(\mathcal{W})) \to (A, \mathcal{V}_f(\mathcal{W}))$ is uniformly continuous.

Proof: Let $V \in \mathcal{V}_f(\mathcal{W})$. By R32.2.15i and R32.2.9i, there is $n \in \mathbb{N}$ and $W \in \mathcal{W}$ such that $(f^n \times f^n)[W] \subseteq V$. The hypothesis implies that $(g \times g)^{-1} = g^{-1} \times g^{-1}$, and g^{-1} commutes with f and so with the repeated composition f^n. Thus

$$(g \times g)^{-1}[(f^n \times f^n)[W] = (f^n \times f^n)[(g \times g)^{-1})[[W]] \subseteq (g \times g)^{-1}[V].$$

Since g is uniformly continuous by hypothesis, $(g \times g)^{-1}[W] \in \mathcal{W}$ and so $(g \times g)^{-1}[V]$ is in $\text{Im}_{f^n}(\mathcal{W}) \subseteq \mathcal{V}_f(\mathcal{W})$. The conlusion now follows.

Corollary R34.2.2 Let $m, n \in \mathbb{N}$ with $m, n \geq 2$.
Then $r_n : (X, \mathcal{V}_m(\mathcal{U})) \to (X, \mathcal{V}_m(\mathcal{U}))$ is uniformly continuous.

Proof: From algebra $r_n \circ r_m = r_m \circ r_n$ and both are uniformly continuous relative to $(X, \mathcal{U})$. The conclusion follows from the previous lemma.

Corollary R34.2.3 Let $m, n \in \mathbb{N}$ with $m, n \geq 2$ and assume $n|m$.
Then $p_n : (X, \mathcal{V}_m(\mathcal{U})) \to (X, \mathcal{V}_m(\mathcal{U}))$ is uniformly continuous.

Proof: Let $m = nk$, where $\in \mathbb{N}$ and, without loss of generality, $k \geq 2$. From algebra $p_n = r_k \circ p_m$. By R32.2.15iii $r_m : (X, \mathcal{V}_m(\mathcal{U})) \to (X, \mathcal{V}_m(\mathcal{U}))$ is a unimorphism and so $p_m = r_m^{-1}$ is also uniformly continuous. By R34.2.2 $r_k : (X, \mathcal{V}_m(\mathcal{U})) \to (X, \mathcal{V}_m(\mathcal{U}))$ is uniformly continuous. Since the composition of uniformly continuous maps is uniformly continuous, the conclusion holds.

Lemma R34.2.4 Let $(A, \mathcal{W})$ be a separated, totally bounded uniform space. Let h be an auto-homeomorphism of $(A, \tau(\mathcal{W}))$ such that h is uniformly continuous relative to $(A, \mathcal{W})$. Let $\mathcal{W}_1$ be a uniformity for A such that $\mathcal{W} \subseteq \mathcal{W}_1$ and $h : (A, \mathcal{W}_1) \to (A, \mathcal{W}_1)$ is a unimorphism. Then $\mathcal{V}_h(\mathcal{W}) \subseteq \mathcal{W}_1$.

Proof: Let $V \in \mathcal{V}_h(\mathcal{W})$. By R32.2.15i there is $n \in \mathbb{N}$ such that $V \in \text{Im}_h^n(\mathcal{W})$, which is $\text{Im}_{h^n}(\mathcal{W})$ by R32.2.9i. By definition R32.2.1 there is $W \in \mathcal{W}$ such that $(h^n \times h^n)[W] \subseteq V$. By hypothesis $W \in \mathcal{W}_1$ and h (and so the repeated composition h^n) is a unimorphism relative to $\mathcal{W}_1$. Thus $(h^n \times h^n)[W]$ and the superset V are in $\mathcal{W}_1$.

The previous lemma could also be expressed by saying $\mathcal{V}_h(\mathcal{W})$ is the smallest uniformity for A which contains $\mathcal{W}$ and makes h a unimorphism.

Corollary R34.2.5 Let $m, n \in \mathbb{N}$ with $m, n \geq 2$ and assume $n|m$.
Then $\mathcal{V}_n(\mathcal{U}) \subseteq \mathcal{V}_m(\mathcal{U})$.

Proof: By R34.2.2 $r_n : (X, \mathcal{V}_m(\mathcal{U})) \to (X, \mathcal{V}_m(\mathcal{U}))$ is uniformly continuous and by R34.2.3 $r_n^{-1} = p_n : (X, \mathcal{V}_m(\mathcal{U})) \to (X, \mathcal{V}_m(\mathcal{U}))$ is uniformly continuous. Thus r_n is a unimorphism relative to $\mathcal{V}_m(\mathcal{U})$. Since $\mathcal{U} \subseteq \mathcal{V}_m(\mathcal{U})$, R32.2.4 can be applied to obtain the conclusion.

Corollary R34.2.6 $\mathcal{V} = \cup\{\mathcal{V}_n(\mathcal{U}) : n \geq 2\}$.

Proof: The supremum automatically contains the union. A basic entourage for the supremum is a finite intersection of the form $\cap_{i=1}^{t} V_{n(i)}$ where $V_{n(i)} \in \mathcal{V}_{n(i)}(\mathcal{U})$ for $1 \leq i \leq t$. Let $m = \Pi_{i=1}^{t} n(i)$. By R34.2.5 $V_{n(i)} \in \mathcal{V}_m(\mathcal{U})$ for any i, as is the finite intersection. Since the union contains every basic entourage and is closed for supersets, $\mathcal{V}$ is contained in the union.

Proposition R34.2.7 Let $n \in \mathbb{N}$ with $n \geq 2$. Then $r_n : (X, \mathcal{V}) \to (X, \mathcal{V})$ is a unimorphism.

Proof: By R34.2.2 $r_n : (X, \mathcal{V}_m(\mathcal{U})) \to (X, \mathcal{V}_m(\mathcal{U}))$ is uniformly continuous for every $m \in \mathbb{N}$ with $m \geq 2$. It follows that r_n is also uniformly continuous relative to the supremum, $\mathcal{V}$. Now let $V \in \mathcal{V}$. By R34.2.6 there is j such that $V \in \mathcal{V}_j(\mathcal{U})$. By R34.2.3 $p_n : (X, \mathcal{V}_{nj}(\mathcal{U})) \to (X, \mathcal{V}_{nj}(\mathcal{U}))$ is uniformly continuous. Since $\mathcal{V}_j(\mathcal{U}) \subseteq \mathcal{V}_{nj}(\mathcal{U})$, by uniform continuity $(p_n \times p_n)^{-1}[V] \in \mathcal{V}_{nj}(\mathcal{U}) \subseteq \mathcal{V}$. Thus $p_n : (X, \mathcal{V}) \to (X, \mathcal{V})$ is uniformly continuous. Since $r_n^{-1} = p_n$, the conclusion follows.

Corollary R34.2.8 Let $\mathcal{W}$ be a uniformity for X such that $r_n : (X, \mathcal{W}) \to (X, \mathcal{W})$ is uniformly continuous for all $n \in \mathbb{N}$ with $n \geq 2$ and $\mathcal{U} \subseteq \mathcal{W}$. Then $\mathcal{V} \subseteq \mathcal{W}$.

Proof: By R34.2.4 $\mathcal{V}_n(\mathcal{U}) \subseteq \mathcal{W}$ for all $n \in \mathbb{N}$ with $n \geq 2$, i.e., $\mathcal{W}$ is an upper bound of the collection. Since $\mathcal{V}$ is the least upper bound, $\mathcal{V} \subseteq \mathcal{W}$.

In other words, $\mathcal{V}$ is the smallest uniformity containing $\mathcal{U}$ and making every r_n a unimorphism.

Finally it will be shown that the separated totally bounded uniformity $\mathcal{V}$ does not correspond to the class of the Stone-Čech compactification.

For the rest of this section $g : X \to X$ will denote the exponential function restricted to X, i.e., $g(x) = e^x$ and W will denote the element of $\mathcal{U}$ described in the proof of R34.1.6: Let $B_1 = [0, 2] \cup (\cup_{i=1}^{\infty}[2i + 1, 2i + 2]$ and $B_2 = [0, \infty) - B_1 = \cup_{i=1}^{\infty}(2i, 2i + 1)$. By definition $\{V_{0.1}[B_1], V_{0.1}[B_2]\}$ is a $\mathcal{U}_a$-proximal cover and so an element of $\mathcal{U}$ can be defined by $W = (V_{0.1}[B_1] \times V_{0.1}[B_1]) \cup (V_{0.1}[B_2] \times V_{0.1}[B_2])$.

It will be shown that $(g \times g)^{-1}[W] \notin \mathcal{V}$. The following calculation will be needed.

Lemma R34.2.9 Let k be a fixed element of $\mathbb{N}$.
Then $\lim_{n\to\infty}([\ln(2n + 1.5)]^k - [\ln(2n + 0.5)]^k) = 0$.

Proof: By the Mean Value Theorem

$$[\ln(2n + 1.5)]^k - [\ln(2n + 0.5)]^k = \frac{k[\ln(m)]^{k-1}}{m}(\ln(2n + 1.5) - \ln(2n + 0.5)),$$

where m is between $2n + 1.5$ and $2n + 0.5$. As $n \to \infty$, $m \to \infty$. By induction and l'Hôpital's rule, $\lim_{m\to\infty} \frac{k[\ln(m)]^{k-1}}{m} = 0$. The second factor on the right is $\ln(\frac{2n+1.5}{2n+0.5})$, which tends to $\ln 1$ as $n \to \infty$. The conclusion follows.

Note that two binary operations are being used here, function composition and multiplication of real numbers, and so exponents need to be interpreted appropriately. In the previous lemma k is a multiplicative exponent, while in the next lemma r_m^j refers to repeated composition. Both might occur in the same sentence, e.g., $r_m^j(x)$ is the m^jth root in $[0,\infty)$ of x.

Lemma R34.2.10 $(g\times g)^{-1}[W]\notin\mathcal{V}$.

Proof: Suppose $(g\times g)^{-1}[W]\in\mathcal{V}$. By R32.4.6 it is in $\mathcal{V}_m(\mathcal{U})$ for some m. By R32.2.9i and R32.2.15i $(g\times g)^{-1}[W]$ is in $\mathrm{Im}_{r_m^j}(\mathcal{U})$ for some j. By the definitions of $\mathcal{U}$ and an image uniformity, there is $\{A_1,\ldots,A_t\}$, a finite cover of X, and $\delta>0$ such that, for $W_1=\cup_{s=1}^t(V_\delta[A_s]\times V_\delta[A_s])$, $(r_m^j\times r_m^j)[W_1]\subseteq(g\times g)^{-1}[W]$. $\{A_s\cap B_i:1\le s\le t,1\le i\le 2\}$ is also a finite cover of X. For $W_2=\cup\{(V_\delta[A_s\cap B_i]\times V_\delta[A_s\cap B_i]):1\le s\le t,1\le i\le 2\}$, $W_2\in\mathcal{U}$ and $(r_m^j\times r_m^j)[W_2]\subseteq(r_m^j\times r_m^j)[W_1]$. By R34.2.9 there is $N\in\mathbb{N}$ such that $[\ln(2N+1.5)]^{m^j}-[\ln(2N+0.5)]^{m^j}<\delta$ and so $([\ln(2N+1.5)]^{m^j},[\ln(2N+0.5)]^{m^j})\in V_\delta$. Pick s,i so that $[\ln(2N+1.5)]^{m^j}\in A_s\cap B_i$. Then $([\ln(2N+1.5)]^{m^j},[\ln(2N+0.5)]^{m^j})\in W_2$. Because $r_m^j(x)$ is the m^jth root, $(r_m^j\times r_m^j)[W_2]\subseteq(g\times g)^{-1}[W]$, and the exponential and natural logarithm are inverse functions, $(2N+1.5,2N+0.5)\in W$. Clearly, $2N+1.5$ is in B_1 but not $V_{0.1}[B_2]$ and $2N+0.5$ is in B_2 but not $V_{0.1}[B_1]$. By the definition of W, $(2N+1.5,2N+0.5)\notin W$, a contradiction.

Corollary R34.2.11 The function g is not uniformly continuous from $(X,\mathcal{V})$ to $(X,\mathcal{V})$.

Proof: W is in $\mathcal{V}$ but $(g\times g)^{-1}[W]\notin\mathcal{V}$.

Corollary R34.2.12 Let (Y,f) be a T_2-compactification of $(X,\tau(\mathcal{V}))$ in the class corresponding to $\mathcal{V}$. Then g does not have a continuous extension from Y to Y.

Proof: By R7.Add.7 g extends continuously to Y if and only if it is uniformly continuous from $(X,\mathcal{V})$ to $(X,\mathcal{V})$. That and the corollary imply that g does not extend.

Corollary R34.2.13 Let (Y,f) be a T_2-compactification of $(X,\tau(\mathcal{V}))$ in the class corresponding to $\mathcal{V}$. Then (Y,f) is not equivalent to the Stone-Čech compactification.

Proof: Every continuous map from $(X,\tau(\mathcal{V}))$ to $(X,\tau(\mathcal{V}))$ extends continuously to any compactification equivalent to the Stone-Čech compactification. Since $\tau(\mathcal{V})$ is the usual topology for X, g is continuous. By the previous corollary g does not extend continuously to Y. The conclusion follows.

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