Extensions of Auto-Homeomorphisms

In this section the question of whether an auto-homeomorphism of a $T_{3\frac{1}{2}}$ space extends to an auto-homeomorphism of a given compactification is considered. The question of whether it might extend continuously but not injectively is raised and answered positively in an added subsection.

General Facts

Definition R32.1.1 Let (X, τ) be a topological space. An auto-homeomorphism of (X, τ) is an onto homeomorphism $h : (X, \tau) \to (X, \tau)$.

For emphasis, the definition assumes $h[X] = X$.

In what follows an extention is understood in the sense of [5]: Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let (Y, f) be a T_2-compactification of (X, τ). A continuous map $h : X \to X$ extends to a continuous $H : Y \to Y$ provided $H \circ f = f \circ h$. If such an extension H exists, it must be unique.

Theorem R32.1.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Let (Y, f) be a T_2-compactification of (X, τ). Let $\mathcal{U}$ be the separated, totally bounded uniformity corresponding to the compactification class of (Y, f). Let $h : X \to X$ be an auto-homeomorphism. Then the following are equivalent

i) h has a continuous, one-to-one extension to Y.

ii) h extends to an auto-homeomorphism of Y.

iii) $h : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a unimorphism.

Proof: Since $f[X]$ is dense in Y, a continuous extension must be onto. The compactness of Y shows that i) implies ii). Now assume h extends to an auto-homeomorphism $H : Y \to Y$. With these hypotheses the equation $H \circ f = f \circ h$ implies $f \circ h^{-1} = H^{-1} \circ f$, i.e., H^{-1} is a continuous extension of h^{-1}. By R7.Add.7 h and h^{-1} are uniformly continuous and so h is a unimorphism. Finally assume iii). By R7.1.3 h extends continuously to $H : Y \to Y$ and h^{-1} to $G : Y \to Y$. It is easy to check that $G \circ H$ agrees with the identity map restricted to $f[X]$. Since the maps are continuous, $f[X]$ is dense, and Y is T_2, $G \circ H$ is the identity on Y. Thus H is one-to-one and i) holds.

The question of whether an auto-homeomorphism can extend continuously to a non-injective map is not answered by the above.

The next corollary, which might be regarded as obvious, can be easily derived from the definition of equivalence. What follows is a uniformity-based argument.

Corollary R32.1.3 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Let (Y, f) and (Z, g) be equivalent T_2-compactifications of (X, τ). Let h be an auto-homeomorphism of X. Then h extends to an auto-homeomorphism of Y if and only if h extends to an auto-homeomorphism of Z.

Proof: The given equivalent compactifications are in the same compactification class, which corresponds to a unique separated totally bounded uniformity $\mathcal{U}$. The question of whether $h : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a unimorphism does not depend on the class representative.

Next two extreme cases are dealt with.

Proposition R32.1.4 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Every auto-homeomorphism of X extends to an an auto-homeomorphism of βX, the Stone-Čech compactification of X.

Proof: Let h be an auto-homeomorphism of X. Both h and h^{-1} are continuous and so by the characterizing property of Stone-Čech compactifications, both extend to βX. Let

$\mathcal{U}_M$ be the uniformity corresponding to the class of the Stone-Čech compactification of X. By R7.Add.7 both h and h^{-1} are uniformly continuous from $(X,\mathcal{U}_M)$ to itself, i.e., h is a unimorphism. The conclusion now follows from R32.1.2.

The next proof implicitly uses R32.1.3 by using a specific representative from the class of the one-point compactification.

Proposition R32.1.5 Let (X, τ) be a non-compact, locally compact, T_2 space. Every auto-homeomorphism of X extends to an an auto-homeomorphism the one-point compactification of X.

Proof: Let h be an auto-homeomorphism of X and let $X^+ = X \cup \{x_0\}$, where x_0 is some point not in X. Let τ^+ be the topology of the one-point compactification, i.e., $O \in \tau^+$ if and only if $O \cap X \in \tau$ and $x_0 \in O$ implies $X - O$ is compact. The embedding ι^+ is inclusion, i.e., $\iota^+(x) = x$. Define $H : X^+ \to X^+$ by $H(x_0) = x_0$ and $H|_X = h$. Clearly H is a bijection and $H \circ \iota^+ = \iota^+ \circ h$. Let $O \in \tau^+$. If $x_0 \notin O$, then $H^{-1}[O] = h^{-1}[O]$ which is in τ by the continuity of h. If $x_0 \in O$, then $O = G \cup \{x_0\}$, where $G \in \tau$ and $X - G$ is compact. $H^{-1}[O] = h^{-1}[G] \cup \{x_0\}$. $H^{-1}[O] \cap X = h^{-1}[G]$ which is in τ. $X - H^[O] = X - h^{-1}[G] = h^{-1}[X - O]$, which is compact since h^{-1} is continuous. Thus $H^{-1}[O] \in \tau^+$ and so H is continuous. By R32.1.2 H is an auto-homeomorphism of X^+ which extends h.

Proposition R32.1.6 Let (X, τ) a $T_{3\frac{1}{2}}$ space and let h be an auto-homeomorphism of X. Let Δ be a non-empty set. Let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ be a collection of T_2 compactifications of (X, τ). Assume, for every $\alpha \in \Delta$, h has a an extension to an auto-homeomorphism H_α of Y_α. Then there exists an auto-homeomorphism H of $\vee Y_\alpha$, which is an extension of h.

Proof: By R7.1.5 both h and h^{-1} have a continuous extensions, H and G respectively, to $\vee Y_\alpha$. It is easy to check that $G \circ H$ is the identity map on $f[X]$, where f is the embedding from X into $\vee Y_\alpha$. Again, because of density and the image space being T_2, $G \circ H$ is the identity map on $\vee Y_\alpha$. Thus H is one-to-one and so an auto-homeomorphism by R32.1.2.

Image Uniformities

In what follows a given permutation of a set is not assumed to have any continuity or uniform continuity properties unless explicitly stated.

Definition R32.2.1 Let $(X,\mathcal{U})$ be a uniform space and let σ be a permutation of X. $\mathrm{Im}_\sigma(\mathcal{U})$ is defined to be $\{S \subseteq X \times X : (\sigma \times \sigma)[U] \subseteq S \text{ for some } U \in \mathcal{U}\}$.

The first lemma records some expected facts, which depend heavily on the bijectivity of the given map.

Lemma R32.2.2 Let $(X,\mathcal{U})$ be a uniform space and let σ be a permutation of X. Then

i) $\mathrm{Im}_\sigma(\mathcal{U})$ is a uniformity on X.

ii) If $\mathcal{U}$ is separated, then so is $\mathrm{Im}_\sigma(\mathcal{U})$.

iii) If $\mathcal{U}$ is totally bounded, then so is $\mathrm{Im}_\sigma(\mathcal{U})$.

iv) $\sigma : (X,\mathcal{U}) \to (X, \mathrm{Im}_\sigma(\mathcal{U})))$ is a unimorphism.

Proof: For i): The required properties transfer from $\mathcal{U}$ as follows. Each element of $\mathrm{Im}_\sigma(\mathcal{U})$ contains the diagonal of $X \times X$ because σ is onto and each element of $\mathcal{U}$ does. The superset property is clear from the definition and the symmetric property follows from $(\sigma \times \sigma)[U^{-1}] = ((\sigma \times \sigma)[U])^{-1}$. Because σ is one-to-one, $(\sigma \times \sigma)[U \cap V] = (\sigma \times \sigma)[U] \cap (\sigma \times \sigma)[V]$ and so the intersection property holds. Finally, since σ is one-to-one,

$(\sigma\times\sigma)[V\circ V] = (\sigma\times\sigma)[V]\circ(\sigma\times\sigma)[V]$ so that the triangle property holds. By definition P2.1 $\mathrm{Im}_\sigma(\mathcal{U})$ is a uniformity. Now assume $\mathcal{U}$ is separated and let $a, b \in X$ with $a \neq b$. Because σ is an onto function, there are $c, d \in X$ with $c \neq d$ such that $\sigma(c) = a$ and $\sigma(d) = b$. There is $U \in \mathcal{U}$ with $(c, d) \notin U$. Because σ is one-to-one, $(a, b) \notin (\sigma \times \sigma)[U]$. Thus ii) holds. For the permutation σ, $U \in \mathcal{U}$, and $x_1, \ldots, x_n \in X$, $\sigma[\cup_{i=1}^n U[x_i]] = \cup_{i=1}^n (\sigma \times \sigma)[U][\sigma(x_i)]$. This implies part iii). Lastly, $(\sigma \times \sigma)^{-1}[(\sigma \times \sigma)[U]) = U$ and so $\sigma : (X,\mathcal{U}) \to (X, \mathrm{Im}_\sigma(\mathcal{U}))$ is uniformly continuous. For $U \in \mathcal{U}$, $(\sigma^{-1} \times \sigma^{-1})^{-1}[U] = (\sigma \times \sigma)[U]$, which is in $\mathrm{Im}_\sigma(\mathcal{U}))$ by definition, and so $\sigma^{-1} : (X, \mathrm{Im}_\sigma(\mathcal{U})) \to (X,\mathcal{U})$ is also uniformly continuous. Thus iv) holds.

Lemma R32.2.3 Let $(X,\mathcal{U})$ be a uniform space and let σ be a permutation of X. Assume $\sigma : (X,\mathcal{U}) \to (X,\mathcal{U})$ is uniformly continuous. Then $\mathcal{U} \subseteq \mathrm{Im}_\sigma(\mathcal{U}))$.

Proof: Let $U \in \mathcal{U}$. By hypothesis $(\sigma \times \sigma)^{-1}[U]$ is also in $\mathcal{U}$. By definition $(\sigma \times \sigma)[(\sigma \times \sigma)^{-1}[U] = U$ is in $\mathrm{Im}_\sigma(\mathcal{U}))$.

Corollary R32.2.4 Let (X, τ) a $T_{3\frac{1}{2}}$ space and let h be an auto-homeomorphism of X. Let (Y, f) be a T_2-compactification of (X, τ). Let $\mathcal{U}$ be the separated, totally bounded uniformity corresponding to the compactification class of (Y, f). Then h extends to an auto-homeomorphism of Y if and only if $\mathrm{Im}_h(\mathcal{U}) = \mathcal{U}$.

Proof: The sufficiency of the condition follows from R32.1.2 and R32.2.2iv. For necessity, if h extends to an auto-morphism, by R32.1.2 $h : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a unimorphism and so, for every $U \in \mathcal{U}$, $(h \times h)[U]$ is in $\mathcal{U}$, which implies $\mathrm{Im}_\sigma(\mathcal{U}) \subseteq \mathcal{U}$. This and the previous lemma show $\mathrm{Im}_\sigma(\mathcal{U}) = \mathcal{U}$.

Lemma R32.2.5 Let $(X,\mathcal{U})$ be a uniform space and let σ be a permutation of X. Assume $\sigma : (X, \tau(\mathcal{U})) \to (X, \tau(\mathcal{U}))$ is open. Then $\tau(\mathrm{Im}_\sigma(\mathcal{U})) \subseteq \tau(\mathcal{U})$.

Proof: Let $b \in G \in \tau(\mathrm{Im}_\sigma(\mathcal{U}))$ and let $a = \sigma^{-1}(b)$. There is $U \in \mathcal{U}$ such that $(\sigma \times \sigma)[U][b] \subseteq G$. There is $O \in \tau(\mathcal{U})$ such that $a \in O \subseteq U[a]$. By hypothesis $\sigma[O] \in \tau(U)$ and clearly $b \in \sigma[O]$. Then $\sigma[O] \subseteq G$ as follows. Let $c \in \sigma[O]$ so that $\sigma^{-1}(c) \in O \subseteq U[a]$. Thus $(a, \sigma^{-1}(c)) \in U$ so that $(b, c) = (\sigma(a), \sigma(\sigma^{-1}(c))) \in (\sigma \times \sigma)[U]$ and $c \in (\sigma \times \sigma)[U][b]$, which is contained in G. This shows that G is a $\tau(\mathcal{U})$-neighborhood of all its points, i.e., $G \in \tau(U)$.

Lemma R32.2.6 Let $(X,\mathcal{U})$ be a uniform space and let σ be a permutation of X. Assume $\sigma : (X, \tau(\mathcal{U})) \to (X, \tau(\mathcal{U}))$ is continuous. Then $\tau(\mathcal{U}) \subseteq \tau(\mathrm{Im}_\sigma(\mathcal{U}))$.

Proof: Let $x \in O \in \tau(\mathcal{U})$. Since $\sigma^{-1}[O] \in \tau(\mathcal{U})$ by continuity, there is $U \in \mathcal{U}$ with $U[\sigma^{-1}(x)] \subseteq \sigma^{-1}[O]$. By definition $(\sigma \times \sigma)[\mathcal{U}] \in \mathrm{Im}_\sigma(\mathcal{U})$. Claim: $(\sigma \times \sigma)[U][x] \subseteq O$. Let $t \in (\sigma \times \sigma)[U][x]$ so that $(\sigma^{-1}(x), \sigma^{-1}(t)) \in U$ and $\sigma^{-1}(t) \in U[\sigma^{-1}(x)]$, which is contained in $\sigma^{-1}[O]$. Then $t \in O$ and the claim holds. Thus $O \in \tau(\mathrm{Im}_\sigma(\mathcal{U}))$

Corollary R32.2.7 Let $(X,\mathcal{U})$ be a uniform space. Let h be an auto-homeomorphism of $(X, \tau(\mathcal{U}))$. Then $\tau(\mathrm{Im}_h(\mathcal{U})) = \tau(\mathcal{U})$.

Proof: This is immediate from R32.2.5 and R32.2.6.

The following results describe a construction possible based an auto-homeomorphism. It would be of interest if the auto-homeomorphism extends continuously to a compactification, with the extension not being one-to-one. In what follows, for an map $m : X \to X$ and positive integer n, m^n denotes repeated composition.

Definition R32.2.8 Let $(X,\mathcal{U})$ be a uniform space. Let σ be a permutation of X. For $n \in \mathbb{N}$, define $\mathrm{Im}_\sigma^n(\mathcal{U})$ inductively by $\mathrm{Im}_\sigma^1(\mathcal{U}) = \mathrm{Im}_\sigma(\mathcal{U})$ and $\mathrm{Im}_\sigma^{n+1}(\mathcal{U}) = \mathrm{Im}_\sigma(\mathrm{Im}_\sigma^n(\mathcal{U}))$.

Lemma R32.2.9 Let $(X,\mathcal{U})$ be a uniform space. Let h be an auto-homeomorphism of $(X,\tau(\mathcal{U}))$ with $h:(X,\mathcal{U})\to(X,\mathcal{U})$ uniformly continuous. Let $n\in\mathbb{N}$. Then

i) $\mathrm{Im}_h^n(\mathcal{U})=\mathrm{Im}_{h^n}(\mathcal{U})$.

ii) $h:(X,\mathrm{Im}_h^n(\mathcal{U}))\to(X,\mathrm{Im}_h^n(\mathcal{U}))$ is uniformly continuous.

iii) $\mathrm{Im}_h^n(\mathcal{U})\subseteq\mathrm{Im}_h^{n+1}(\mathcal{U})$

iv) $h:(X,\mathrm{Im}_h^n(\mathcal{U}))\to(X,\mathrm{Im}_h^{n+1}(\mathcal{U}))$ is a unimorphism.

v) $h^n:(X,\mathcal{U})\to(X,\mathrm{Im}_h^n(\mathcal{U}))$ is a unimorphism.

vi) $\tau(\mathrm{Im}_h^n(\mathcal{U}))=\tau(\mathcal{U})$.

vii) If $\mathcal{U}$ is separated and totally bounded, then $\mathrm{Im}_h^n(\mathcal{U})$ is seprated and totally bounded.

Proof: Part i) follows by induction from the definitions and the easy checked fact that, for $S\subseteq X\times X$, $(h^{n+1}\times h^{n+1})[S]=(h\times h)[(h^n\times h^n)[S]]$. For part ii) proceed by induction again. If $n=1$, by R32.2.2iv $h:(X,\mathcal{U})\to(X,\mathrm{Im}_\sigma^1(\mathcal{U}))$ is uniformly continuous. With the hypothesis, R32.2.3 shows that $\mathcal{U}\subseteq\mathrm{Im}_\sigma^1(\mathcal{U})$ and so h is also uniformly continuous with the larger domain uniformity. Now assume $h:(X,\mathrm{Im}_h^n(\mathcal{U}))\to(X,\mathrm{Im}_h^n(\mathcal{U}))$ is uniformly continuous. A similar argument using the previous definition, R32.2.2iv, and R32.2.3 shows $h:(X,\mathrm{Im}_h^{n+1}(\mathcal{U}))\to(X,\mathrm{Im}_h^{n+1}(\mathcal{U}))$ is also uniformly continuous. Part iii) follows from ii) and R32.2.3, and part iv) from the definition and R32.2.2iv. The last three parts proceed by induction. When $n=1$, v) restates R32.2.2iv. Now assume $h^n:(X,\mathcal{U})\to(X,\mathrm{Im}_h^n(\mathcal{U}))$ is a unimorphism. Since the composition of two unimorphisms is again a unimorphism, by part iv) and the induction hypothesis $h^{n+1}=h\circ h^n$ is a unimorphism from $(X,\mathcal{U})$ to $(X,\mathrm{Im}_h^{n+1}(\mathcal{U}))$. Thus v) holds. When $n=1$, part vi) restates R32.2.7. In the induction step $\tau(\mathrm{Im}_h^n(\mathcal{U})=\tau(\mathcal{U})$ so that R32.2.7 and the definition yield $\tau(\mathrm{Im}_h^{n+1}(\mathcal{U}))=\tau(\mathrm{Im}_h^n(\mathcal{U}))$, which is $\tau(\mathcal{U})$. Part vii) follows from the second and third parts of R32.2.2, the definition, and a similarly routine induction.

Lemma R32.2.10 Let $(X,\mathcal{U})$ be a uniform space. Let h be an auto-homeomorphism of $(X,\tau(\mathcal{U}))$ with $h:(X,\mathcal{U})\to(X,\mathcal{U})$ uniformly continuous. Let $j\in\mathbb{N}$ and assume $\mathrm{Im}_h^{j+1}(\mathcal{U})=\mathrm{Im}_h^j(\mathcal{U})$. Then $\mathrm{Im}_h^n(\mathcal{U})=\mathrm{Im}_h^1(\mathcal{U})$ for all $n\in\mathbb{N}$.

Proof: First note by induction that $\mathrm{Im}_h^n(\mathcal{U})=\mathrm{Im}_h^j(\mathcal{U})$ for all $n\geq j$. By hypothesis this holds when $n=j$. If it is true for some $n\geq j$, $\mathrm{Im}_h^{n+1}(\mathcal{U})=\mathrm{Im}_h(\mathrm{Im}_h^n(\mathcal{U}))=\mathrm{Im}_h(\mathrm{Im}_h^j(\mathcal{U}))=\mathrm{Im}_h^{j+1}(\mathcal{U})=\mathrm{Im}_h^j(\mathcal{U})$ and so the claim holds. Now let t be the smallest in $\{n:\mathrm{Im}_h^{n+1}(\mathcal{U})=\mathrm{Im}_h^n(\mathcal{U})\}$, which is non-empty by hypothesis. If $t=1$, the initial observation shows that the conclusion holds. Suppose $t>1$ and let $V\in\mathrm{Im}_h^t(\mathcal{U})$. $(h\times h)[V]$ is in $\mathrm{Im}_h^{t+1}(\mathcal{U})=\mathrm{Im}_h^t(\mathcal{U})$ and so by R32.2.9i there is $U\in\mathcal{U}$ such that $(h^t\times h^t)[U]\subseteq(h\times h)[V]$. By applying $(h\times h)^{-1}$ one obtains $(h^{t-1}\times h^{t-1})[U]\subseteq V$. Thus $V\in\mathrm{Im}_h^{t-1}(\mathcal{U})$ and $\mathrm{Im}_h^t(\mathcal{U})\subseteq\mathrm{Im}_h^{t-1}(\mathcal{U})$. By that, R32.2.9ii, and R32.2.3 $\mathrm{Im}_h^{t-1}(\mathcal{U})=\mathrm{Im}_h^t(\mathcal{U})$, which contradicts the assumption that $t>1$.

Corollary R32.2.11 Let $(X,\mathcal{U})$ be a uniform space and assume that h is an auto-homeomorphism of $(X,\tau(\mathcal{U}))$ with $h:(X,\mathcal{U})\to(X,\mathcal{U})$ uniformly continuous. Let $j\in\mathbb{N}$ with $\mathrm{Im}_h^{j+1}(\mathcal{U})=\mathrm{Im}_h^j(\mathcal{U})$. Then $\mathrm{Im}_h^1(\mathcal{U})=\mathcal{U}$.

Proof: Let $W\in\mathrm{Im}_h^1(\mathcal{U})$. There is $V\in\mathcal{U}$ such that $(h\times h)[V]\subseteq W$. By definition $(h\times h)[(h\times h)[V]]$ is in $\mathrm{Im}_h^2(\mathcal{U})$, which equals $\mathrm{Im}_h^1(\mathcal{U})$ by R32.2.10. Thus there is $U\in\mathcal{U}$ with $(h\times h)[U]\subseteq(h\times h)[(h\times h)[V]$. By applying $(h\times h)^{-1}$ one obtains$U\subseteq(h\times H)[V]\subseteq W$ and so $\mathrm{Im}_h^1(\mathcal{U})\subseteq\mathcal{U}$. By R32.2.3 the conclusion follows.

In the next corollary the compactifications and extensions exist based on the above, [3], and [5]. With the stated hypotheses, R32.2.9vi shows that both the compactifications mentioned are compactifications of the same $T_{3\frac{1}{2}}$ topology.

Corollary R32.2.12 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space. Let h be an auto-homeomorphism of $(X,\tau(\mathcal{U}))$ with $h:(X,\mathcal{U})\to(X,\mathcal{U})$ uniformly continuous. Let $\mathcal{U}$ correspond to the compactification class of the T_2-compactification (Y,f). Let n be in $\mathbb{N}$ and let $\text{Im}_h^n(\mathcal{U})$ correspond to the compactification class of the T_2-compactification (Y_n,f_n). If the extension of h from Y to Y is not one-to-one, then the extension of h from Y_n to Y_n is also not one-to-one.

Proof: Assume the extension of h from Y to Y is not one-to-one. $\text{Im}_h^1(\mathcal{U})\neq\mathcal{U}$ by R32.2.4. By R32.2.11 $\text{Im}_h^{n+1}(\mathcal{U})\neq\text{Im}_h^n(\mathcal{U})$. By R32.2.4 again the extension of h from Y_n to Y_n is also not one-to-one.

Lemma R32.2.13 Let $(X,\mathcal{U})$ be a uniform space. Let $h:(X,\mathcal{U})\to(X,\mathcal{U})$ be uniformly continuous with h an auto-homeomorphism of $(X,\tau(\mathcal{U}))$. Assume $\text{Im}_h^2(\mathcal{U})\neq\text{Im}_h^1(\mathcal{U})$. Let $n\in\mathbb{N}$ with $n\geq 2$. Then $h:(X,\mathcal{U})\to(X,\text{Im}_h^n(\mathcal{U}))$ is not uniformly continuous.

Proof: Deny the conclusion and let $V\in\text{Im}_h^n(\mathcal{U})$. Then $(h\times h)^{-1}[V]\in\mathcal{U}$ and so $(h\times h)[(h\times h)^{-1}[V]=V$ is in $\text{Im}_h^1(\mathcal{U})$. Thus $\text{Im}_h^n(\mathcal{U})\subseteq\text{Im}_h^1(\mathcal{U})$. R32.2.9iii implies $\text{Im}_h^1(\mathcal{U})\subseteq\text{Im}_h^2(\mathcal{U})\subseteq\text{Im}_h^n(\mathcal{U})$. Thus $\text{Im}_h^1(\mathcal{U})=\text{Im}_h^2(\mathcal{U})$, a contradiction.

Definition R32.2.14 Let $(X,\mathcal{U})$ be a uniform space. Let $h:(X,\mathcal{U})\to(X,\mathcal{U})$ be uniformly continuous with h an auto-homeomorphism of $(X,\tau(\mathcal{U}))$. The uniformity $\mathcal{V}_h(\mathcal{U})$ is defined as $\vee\{\text{Im}_h^n(\mathcal{U}):n\in\mathbb{N}\}$.

Lemma R32.2.15 Let $(X,\mathcal{U})$ be a uniform space. Let h be an auto-homeomorphism of $(X,\tau(\mathcal{U}))$ with $h:(X,\mathcal{U})\to(X,\mathcal{U})$ uniformly continuous. Then

- i) $\mathcal{V}_h(\mathcal{U})=\cup\{\text{Im}_h^n(\mathcal{U}):n\in\mathbb{N}\}$.
- ii) $\tau(\mathcal{V}_h(\mathcal{U}))=\tau(\mathcal{U})$.
- iii) $h:(X,\mathcal{V}_h(\mathcal{U}))\to(X,\mathcal{V}_h(\mathcal{U}))$ is a unimorphism.
- iv) If $h:(X,\mathcal{U})\to(X,\mathcal{U})$ is a unimorphism, then $\mathcal{V}_h(\mathcal{U})=\mathcal{U}$.
- v) If $\mathcal{U}$ is separated and totally bounded, $\mathcal{V}_h(\mathcal{U})$ is also separated and totally bounded.

Proof: Since $\mathcal{V}_h(\mathcal{U})$ is an upper bound, $\cup\{\text{Im}_h^n(\mathcal{U}):n\in\mathbb{N}\}\subseteq\mathcal{V}_h(\mathcal{U})$. By R32.2.9iii these uniformities form an ascending chain and so checking that $\cup\{\text{Im}_h^n(\mathcal{U}):n\in\mathbb{N}\}$ is a uniformity is routine. Therefore it must be the least upper bound, i.e., i) holds. Part ii) follows from R32.2.9vi and P2.14. For part iii) let $V\in\mathcal{V}_h(\mathcal{U})$. By i) $V\in\text{Im}_h^n(\mathcal{U})$ for some n. If $n=1$, $(h\times h)^{-1}[V]\in\mathcal{U}$ by R32.2.2iv and $\mathcal{U}\subseteq\text{Im}_h^1(\mathcal{U})\subseteq\mathcal{V}_h(\mathcal{U})$ by R32.2.3. If $n\geq 2$, By R32.2.2iv $(h\times h)^{-1}[V]$ is in $\text{Im}_h^{n-1}(\mathcal{U})\subseteq\mathcal{V}_h(\mathcal{U})$. Thus h is uniformly continuous. By definition $(h\times h)[V]\in\text{Im}_h^{n+1}(\mathcal{U})\subseteq\mathcal{V}_h(\mathcal{U})$ and h is given to be a permutation. Thus iii) holds. For part iv) assume the unimorphism, which implies $(h\times h)[U]\in\mathcal{U}$ for all $U\in\mathcal{U}$, i.e., $\text{Im}_h^1(\mathcal{U})\subseteq\mathcal{U}$. That and R32.2.3 show that $\text{Im}_h^1(\mathcal{U})=\mathcal{U}$. A routine induction shows $\text{Im}_h^n(\mathcal{U})=\mathcal{U}$ for all n and so by part i) $\mathcal{V}_h(\mathcal{U})=\mathcal{U}$. Finally, assume $\mathcal{U}$ is separated and totally bounded. Since $\mathcal{U}\subseteq\text{Im}_h^1(\mathcal{U})\subseteq\mathcal{V}_h(\mathcal{U})$, the larger uniformity is separated because the smaller is. By R32.2.9vii and P2.13, $\mathcal{V}_h(\mathcal{U})$ is totally bounded.

Corollary R32.2.16 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space. Let h be an auto-homeomorphism of $(X,\tau(\mathcal{U}))$ with $h:(X,\mathcal{U})\to(X,\mathcal{U})$ uniformly continuous.

Let $\mathcal{U}$ correspond to the compactification class of the T_2-compactification (Y, f). For n in $\mathbb{N}$ let $\mathrm{Im}_h^n(\mathcal{U})$ correspond to the compactification class of the T_2-compactification (Y_n, f_n). Let $\mathcal{V}_h(\mathcal{U})$ correspond to the T_2-compactification (Z, g). Then the class of (Z, g) acts as the supremum of the classes of the (Y_n, f_n).

Proof: Since $\mathcal{V}_h(\mathcal{U})$ is defined as the supremum of $\{\mathrm{Im}_h^n(\mathcal{U}) : n \in \mathbb{N}\}$, the conclusion is immediate from R13.1.7.

The previous conclusion could be written more concisely in terms of equivalence, $(Z, g) \approx \vee_{n=1}^{\infty}(Y_n, f_n)$, or loosely as a pseudo-equation, $(Z, g) = \vee_{n=1}^{\infty}(Y_n, f_n)$.

Corollary R32.2.17 Let $(X, \mathcal{U})$ be a separated, totally bounded uniform space. Let h be an auto-homeomorphism of $(X, \tau(\mathcal{U}))$ with $h : (X, \mathcal{U}) \to (X, \mathcal{U})$ uniformly continuous. Let $\mathcal{V}_h(\mathcal{U})$ correspond to the class of the T_2-compactification (Z, g). Then h extends to an auto-homeomorphism of Z.

Proof: This follows from R32.2.15iii and R32.1.2.

Finite-Point Compactifications

Proposition R32.3.1 Let (X, τ) be a locally compact T_2 space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let (Y, f) be a finite-point compactification of (X, τ) and assume h extends continuously to $H : Y \to Y$. Then H is an auto-homeomorphism.

Proof: The extension equation $H \circ f = f \circ h$ and the fact that f and h are one-to-one show that H restricted to $f[X]$ is one-to-one. Since h is onto, $H[f[X]] = f[X]$. Because Y is compact and T_2, the continuity of H and the density of $f[X]$ imply H is onto Y. Since H is a function, i.e. single-valued, H restricted to the finite set $Y - f[X]$ must be onto $Y - f[X]$, and the finiteness implies the restricted map is also on-to-one. It is now easy to check that H is one-to-one on Y. The conclusion follows from R32.1.2.

For convenience, some terminology and facts from [4] will now be summarized. Let (X, τ) be a locally compact T_2 space. For $n \in \mathbb{N}$, an n-star is a set of pairwise disjoint open sets, $\{G_1 \ldots, G_n\}$, such that K, the complement of $\cup_{i=1}^n G_i$ in X, is compact and $K \cup G_i$ is non-compact for each i. Each n-star determines an n-point compactification described as follows. Let $Y = X \cup \{p_1, \ldots, p_n\}$ where $p_i \notin X$ for each i and $i \neq j$ implies $p_i \neq p_j$. Let $\rho = \{O \subseteq Y : O \cap Y \in \tau$ and $p_i \in O$ implies $(X - O) \cap G_i$ has compact closure in $X\}$ and let $f : X \to Y$ be inclusion, $f(x) = x$. With the topology ρ on Y, (Y, f) is an n-point T_2-compactification of (X, τ), which is called the compactification determined by the given n-star. Each finite-point compactification of (X, τ) is equivalent to the compactification determined by an n-star for some n.

Proposition R32.3.2 Let (X, τ) be a locally compact T_2 space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let (Y, f) be the n-point compactification determined by the n-star $\{G_1, \ldots, G_n\}$. Then h extends continuously to Y if and only if there is σ, a permutation of $\{1, \ldots, n\}$, such that $(X - h^{-1}[G_j]) \cap G_{\sigma(j)}$ has compact closure in X for each $1 \leq j \leq n$.

Proof: First assume h extends continuously to H. As in the proof of R32.3.1, H restricted to $\{p_1, \ldots, p_n\}$ is one-to-one and onto $\{p_1, \ldots, p_n\}$. Thus a permutation σ of $\{1, \ldots, n\}$ is induced by $\sigma(k) = i$ provided $H^{-1}(p_k) = p_i$. Now let $1 \leq j \leq n$. By definition of the topology ρ, $G_j \cup \{p_j\} \in \rho$. Thus $H^{-1}[G_j \cup \{p_j\}] = h^{-1}[G_j] \cup \{p_{\sigma(j)}\}$ is also in ρ. $X - H^{-1}[G_j \cup \{p_j\}] = X - h^{-1}[G_j]$ and so, by the definition of ρ, $(X - h^{-1}[G_j]) \cap G_{\sigma(j)}$ has compact closure in X. For the converse assume σ exists and define H by $H(x) = h(x)$

for $x \in X$ and $H(p_{\sigma(k)}) = p_k$. It is easy to check that $H \circ f = f \circ h$, i.e., H extends h. Thus it is sufficient to show that H is continuous. Let $O \in \rho$ and write $O = O_1 \cup S$, where $O_1 = O \cap X$, which is in τ, and $S \subseteq \{p_1, \ldots, p_n\}$. $H^{-1}[O] \cap X = h^{-1}[O_1]$, which is in τ because h is continuous. Let $p_{\sigma(k)} \in H^{-1}[S]$ so that $H(p_{\sigma(k)}) = p_k \in O$. Then $(X - h^{-1}[O_1]) \cap G_{\sigma(k)} \subseteq ((X - h^{-1}[G_k]) \cap G_{\sigma(k)}) \cup ((X - h^{-1}[O_1]) \cap h^{-1}[G_k]))$ as follows: Let $t \in (X - h^{-1}[O_1]) \cap G_{\sigma(k)}$ with $t \notin (X - h^{-1}[G_k]) \cap G_{\sigma(k)}$. It's given that $t \in G_{\sigma(k)}$ so that $t \notin X - h^{-1}[G_k]$, i.e., $t \in h^{-1}[G_k]$. It's also given that $t \in X - h^{-1}[O_1]$ so that $t \in (X - h^{-1}[O_1]) \cap h^{-1}[G_k])$ and the claim is verified. Note that $(X - h^{-1}[O_1]) \cap h^{-1}[G_k]) = h^{-1}[(X - O) \cap G_k]$, which has compact closure in X because $O \in \rho$, $p_k \in O$, and h is a homeomorphism. The other component in the union has compact closure in X by the hypothesis for this part. $(X - h^{-1}[O_1]) \cap G_{\sigma(k)}$, which equals $(X - H^{-1}[O]) \cap G_{\sigma(k)}$, has compact closure in X because closure distributes over finite unions,. Thus $H^{-1}[O] \in \rho$ so that H is continuous as required.

Comment: The previous result seems somehow analogous to R5.1.5, but whether either is a corollary of the other is unclear. Perhaps both are corollaries of some unidentified more general result.

Corollary R32.3.3 Every auto-homeomorphism of $\mathbb{R}$ extends to an auto-homeomorphism of the 2-point compactification of $\mathbb{R}$.

Proof: By R5.1.8 all 2-point compactifications of $\mathbb{R}$ are equivalent and so, by R32.1.3, extendibility can be tested using a convenient representative, say the 2-point compactification determined by the 2-star $\{G_1 = (-\infty, -1), G_2 = (1, \infty)\}$. Let h be an auto-homeomorphism of $\mathbb{R}$. By R32.3.1, it is sufficient to show continuous extendability. The Intermediate Value Theorem implies that h must be strictly increasing or strictly decreasing. In the case that h is strictly increasing h^{-1} is also strictly increasing so that $h^{-1}[G_1] = (-\infty, h^{-1}(-1))$ and $h^{-1}[G_2] = (h^{-1}(1), \infty)$. Let σ be the identity on $\{1, 2\}$. The complement of $h^{-1}[G_1]$ intersected with G_1 is $[h^{-1}(-1), \infty) \cap (-\infty, -1)$, which has compact closure in $\mathbb{R}$. Likewise, $(\mathbb{R} - h^{-1}[G_2]) \cap G_2$ has compact closure in $\mathbb{R}$, and so h extends continuously by R32.3.2. The case of h strictly decreasing is similar, with $\sigma(1) = 2$ and $\sigma(2) = 1$.

Example R32.3.4 Let $\mathbb{N}$ have the discrete topology with (Y, f) the compactification determined by the 2-star $\{G_1, G_2\}$, where G_1 is the set of evens and G_2 is the set of odds. A homeomorphism of $\mathbb{N}$ which does not extend continuously to Y will be constructed. For $j = 1, 2, 3$ let C_j be the equivalence class of j mod 3. Note that $C_2 = \{3i - 1 : i \in \mathbb{N}\}$ and the ith element is even when i is odd and odd when i is even. $C_3 = \{3i : i \in \mathbb{N}\}$ and in that set the ith element is odd when i is odd and even when i is even. Define h by $h(x) = x$ for $x \in C_1$, $h(3i - 1) = 3i$ for $3i - 1 \in C_2$, and $h(3i) = 3i - 1$ for $3i \in C_3$. It is easy to check that h is a permutation of $\mathbb{N}$ and so a homeomorphism of the discrete space. Note that $h^{-1}[G_2]$ contains the infinitely many odd numbers in C_2 and the infinitely many even numbers in C_1. Since $\mathbb{N} - h^{-1}[G_1] = h^{-1}[G_2]$, $h^{-1}[G_2] \cap G_1$ is infinite, and $h^{-1}[G_2] \cap G_2$ is infinite, by R32.3.2 h does not extend continuously to Y.

The last example also illustrates that there is no general relationship between extendibility and the ordering of compactifications: h (as in R32.3.4) extends to auto-homeomorphisms of the larger Stone-Čech compactification (R32.1.4) and of the smaller one-point compactification (R32.1.5).

Wallman Compactifications

In this subsection the possibility of extending auto-homeomorphisms will be considered in the context of Wallman compactifications, i.e., compactifications generated from a normal basis. Notation is as in [2]. The first result is more general than what's needed for the rest of the subsection.

Proposition R32.4.1 Let (X_1, τ_1) and (X_2, τ_2) be $T_{3\frac{1}{2}}$ spaces, let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be a normal bases for (X_1, τ_1) and (X_1, τ_2) respectively, and let $\mathcal{U}_1, \mathcal{U}_2$ be the separated, totally bounded uniformities corresponding to the compactification classes of $(\omega(\mathcal{Z}_1), \iota_{\mathcal{Z}_1})$ and $(\omega(\mathcal{Z}_2), \iota_{\mathcal{Z}_2})$ respectively. Let $f : X_1 \to X_2$ and assume $f^{-1}[Z] \in \mathcal{Z}_1$ for every $Z \in \mathcal{Z}_2$. Then $f : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ is uniformly continuous.

Proof: Let $U \in \mathcal{U}_2$. By R14.1.3 There are $Z_1, \ldots, Z_n$ in $\mathcal{Z}_2$ with $\cap_{i=1}^n Z_i = \emptyset$ and $\cup_{i=1}^n (X - Z_i) \times (X - Z_i) \subseteq U$. In general,

$$(f \times f)^{-1}[\cup_{i=1}^n (X - Z_i) \times (X - Z_i)] = \cup_{i=1}^n (X - f^{-1}[Z_i]) \times (X - f^{-1}[Z_i])$$

and so it is sufficient to show the latter is in $\mathcal{U}_1$. By hypothesis $f^{-1}[Z_i] \in \mathcal{Z}_1$ for each i and $\cap_{i=1}^n f^{-1}[Z_i] = f^{-1}[\cap_{i=1}^n Z_i] = \emptyset$. By R14.1.3 again $\cup_{i=1}^n (X - f^{-1}[Z_i]) \times (X - f^{-1}[Z_i])$ is in $\mathcal{U}_1$ as required.

Corollary R32.4.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\sigma : X \to X$ be a permutation. Let $\mathcal{Z}$ be a normal basis for (X, τ) and let $\mathcal{U}$ be the separated, totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. Assume $\sigma[Z] \in \mathcal{Z}$ for every $Z \in \mathcal{Z}$. Then $\sigma^{-1} : (X, \mathcal{U}) \to (X, \mathcal{U})$ is uniformly continuous.

Proof: Because $(\sigma^{-1})^{-1}[Z] = \sigma[Z]$, this is immediate from R32.4.1.

Corollary R32.4.3 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let h be a permutation of X. Let $\mathcal{Z}$ be a normal basis for (X, τ). Assume for every $Z \in \mathcal{Z}$ both $h[Z]$ and $h^{-1}[Z]$ are in $\mathcal{Z}$. Then h is an auto-homeomorphism of (X, τ) which extends to an auto-homeomorphism of $\omega(\mathcal{Z})$.

Proof: Let $\mathcal{U}$ be the separated, totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. With these assumptions the last two results show that $h : (X, \mathcal{U}) \to (X, \mathcal{U})$ is a unimorphism. Since $\tau(\mathcal{U}) = \tau$, h is an auto-homeomorphism of (X, τ). That it extends follows from R32.1.2.

With the additional assumption that $\mathcal{Z}$ is closed under complementation, which implies that (X, τ) is zero-dimensional and holds for examples such as $\mathcal{Z}_k$, partial converses can be obtained.

Lemma R32.4.4 Let S be a set, let $\{A_\alpha : \alpha \in \Delta\}$ be a non-empty collection of subsets of S, and let B, C be subsets of S. Assume $\cup\{A_\alpha \times A_\alpha : \alpha \in \Delta\} \subseteq (B \times B) \cup (C \times C)$. Then for every α, either $A_\alpha \subseteq B$ or $A_\alpha \subseteq C$.

Proof: Let $\alpha \in \Delta$ and suppose $A_\alpha \not\subseteq B$. Then there exists x such that $x \in A_\alpha - B$. Let $y \in A_\alpha$. By hypothesis $(x, y) \in (B \times B) \cup (C \times C)$. But $x \notin B$ implies $(x, y) \notin B \times B$ and so $(x, y) \in C \times C$. Thus $y \in C$ and $A_\alpha \subseteq C$.

Proposition R32.4.5 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $f : X \to X$. Let $\mathcal{Z}$ be a normal basis for (X, τ) and let $\mathcal{U}$ be the separated, totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. Assume $\mathcal{Z}$ is closed under complementation and $f : (X, \mathcal{U}) \to (X, \mathcal{U})$ is uniformly continuous. Then $f^{-1}[Z] \in \mathcal{Z}$ for every $Z \in \mathcal{Z}$.

Proof: Let $Z \in \mathcal{Z}$. By hypothesis $X - Z$ is also in $\mathcal{Z}$ and so by R14.1.3 $U = (Z \times Z) \cup (X - Z) \times (X - Z)$is in $\mathcal{U}$. By the uniform continuity of f, $(f \times f)^{-1}[U] \in \mathcal{U}$ and so by R14.1.3 again there exist $Z_1, \ldots, Z_n$ in $\mathcal{Z}$ with $\cap_{i=1}^n Z_i = \emptyset$ such that

$$\cup_{i=1}^n (X - Z_i) \times (X - Z_i) \subseteq (f^{-1}[Z] \times f^{-1}[Z]) \cup (X - f^{-1}[Z]) \times (X - f^{-1}[Z]).$$

Clearly $\cup\{X - Z_i : X - Z_i \subseteq f^{-1}[Z]\} \subseteq f^{-1}[Z]$. It is claimed that equality holds. Let $x \in f^{-1}[Z]$. Since $\cap_{i=1}^n Z_i = \emptyset$, there is j with $x \in X - Z_j$. By the previous lemma $X - Z_j$ is contained in one of $f^{-1}[Z], X - f^{-1}[Z]$ and since $x \notin X - f^{-1}[Z]$ it must be that $X - Z_j \subseteq f^{-1}[Z]$. Thus x is in the union and the claimed equality holds. Since $\mathcal{Z}$ is closed under complementation and finite unions, $f^{-1}[Z] \in \mathcal{Z}$.

Corollary R32.4.6 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $f : X \to X$ be a permutation. Let $\mathcal{Z}$ be a normal basis for (X, τ) and let $\mathcal{U}$ be the separated, totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. Assume $\mathcal{Z}$ is closed under complementation and $f^{-1} : (X,\mathcal{U}) \to (X,\mathcal{U})$ is uniformly continuous. Then for every $Z \in \mathcal{Z}$, $f[Z] \in \mathcal{Z}$.

Proof: This follows from R32.4.5 applied to f^{-1} because $(f^{-1})^{-1}[Z] = f[Z]$.

Corollary R32.4.7 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let h be a permutation of X. Let $\mathcal{Z}$ be a normal basis for (X, τ). Assume $\mathcal{Z}$ is closed under complementation and h extends to an auto-homeomorphism of $\omega(\mathcal{Z})$. Then for every $Z \in \mathcal{Z}$, both $h[Z]$ and $h^{-1}[Z]$ are in $\mathcal{Z}$.

Proof: Let $\mathcal{U}$ be the separated, totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. By R32.1.2 $h : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a unimorphism and so the conclusion follows from R32.4.5 and R32.4.6.

Definition R32.4.8 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\sigma : X \to X$ be a permutation. Let $\mathcal{Z}$ be a normal basis for (X, τ). $\mathrm{Zim}_\sigma(\mathcal{Z})$ is defined to be $\{\sigma[Z] : Z \in \mathcal{Z}\}$.

The previous definition will not be used here in its full generality. As in the next result, the focus is on auto-homeomorphisms rather than arbitrary permutations.

Lemma R32.4.9 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}$ be a normal basis for (X, τ). Let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Then $\mathrm{Zim}_h(\mathcal{Z})$ is a normal basis for (X, τ).

Proof: Because h is an auto-homeomorphism, it follows easily that $\mathrm{Zim}_h(\mathcal{Z})$ is a base for the closed subsets of (X, τ). Using only the continuity of h and its bijectivity, the other three requirements for a normal basis in P3.1 transfer routinely from $\mathcal{Z}$ to $\mathrm{Zim}_h(\mathcal{Z})$.

Proposition R32.4.10 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}$ be a normal basis for (X, τ). Let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{U}, \mathcal{V}$ be the separated totally bounded uniformities corresponding to the compactification classes of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ and $(\omega(\mathrm{Zim}_h(\mathcal{Z})), \iota_{\mathrm{Zim}_h(\mathcal{Z})})$ respectively. Then $\mathcal{V} = \mathrm{Im}_h(\mathcal{U})$.

Proof: Because h is a permutation, $h[X - S] = X - h[S]$ for any $S \subseteq X$ and so

$$(*) \cup_{t \in F} (X - h[S_t]) \times (X - h[S_t]) = (h \times h)[\cup_{t \in F}(X - S_t) \times (X - S_t)]$$

for any non-empty collection $\{S_t \subseteq X : t \in F\}$. Now let $V \in \mathcal{V}$. By R14.1.3 and the definition of $\mathrm{Zim}_h(\mathcal{Z})$ there exist $Z_1, \ldots Z_n$ in $\mathcal{Z}$ with $\cap_{i=1}^n h[Z_i] = \emptyset$ such that

$\cup_{i=1}^{n}(X - h[Z_i]) \times (X - h[Z_i]) \subseteq V$. Note that $\cap_{i=1}^{n} Z_i = \emptyset$ and so by R14.1.3 $U = \cup_{i=1}^{n}(X - Z_i) \times (X - Z_i)$ is in $\mathcal{U}$. By (*) $(h \times h)[U] \subseteq V$, i.e., $V \in \mathrm{Im}_h(\mathcal{U})$. Now let $W \in \mathrm{Im}_h(\mathcal{U})$. By R14.1.3 there is a basic entourage in $\mathcal{U}$ determined by $C_1, \ldots, C_j$ in $\mathcal{Z}$ with $\cap_{i=1}^{j} C_i = \emptyset$ such that $(h \times h)[\cup_{i=1}^{j}(X - C_i) \times (X - C_i)] \subseteq W$. Because h is one-to-one, $\cap_{i=1}^{j} h[C_i] = \emptyset$. Each $h[C_i]$ is in $\mathrm{Zim}_h(\mathcal{Z})$. By R14.1.3 again $W_1 = \cup_{i=1}^{j}(X - h[C_i]) \times (X - h[C_i]$ is in $\mathcal{V}$. By (*) $W_1 \subseteq W$ and so $W \in \mathcal{V}$.

This proposition makes available various results from the second subsection, e.g., the following.

Corollary R32.4.11 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}$ be a normal basis for (X, τ). Let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{U}, \mathcal{V}$ be the separated totally bounded uniformities corresponding to the compactification classes of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ and $(\omega(\mathrm{Zim}_h(\mathcal{Z})), \iota_{\mathrm{Zim}_h(\mathcal{Z})})$ respectively. Then $h : (X, \mathcal{U}) \to (X, \mathcal{V})$ is a unimorphism.

Proof: This is immediate from R32.4.10 and R32.2.2iv.

The following inductive definition, which implicitly uses R32.4.9, leads to more connections with image uniformities.

Definition R32.4.12 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{Z}$ be a normal basis for (X, τ). $\mathrm{Zim}_h^1(\mathcal{Z})$ is defined to be $\mathrm{Zim}_h(\mathcal{Z})$. For $n \in \mathbb{N}$, $\mathrm{Zim}_h^{n+1}(\mathcal{Z}) = \mathrm{Zim}_h(\mathrm{Zim}_h^n(\mathcal{Z})$.

Lemma R32.4.13 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{Z}$ be a normal basis for (X, τ). Then

i) For every $n \in \mathbb{N}$, $\mathrm{Zim}_h^n(\mathcal{Z})$ is a normal basis for (X, τ).

ii) For every $n \in \mathbb{N}$, $\mathrm{Zim}_h^n(\mathcal{Z}) = \{h^n[Z] : Z \in \mathcal{Z}\}$.

Proof: Both follow by induction. Part i) holds for $n = 1$ by R32.4.9, which, with the induction hypothesis, also implies $\mathrm{Zim}_h^{n+1}(\mathcal{Z})$ is a normal basis. Part ii) holds when $n = 1$ by definition. The induction step follows easily from $h[h^n[Z]] = h^{n+1}[Z]$.

Proposition R32.4.14 Let $n \in \mathbb{N}$. Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}$ be a normal basis for (X, τ). Let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{U}, \mathcal{V}_n$ be the separated totally bounded uniformities corresponding to the compactification classes of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ and $(\omega(\mathrm{Zim}_h^n(\mathcal{Z})), \iota_{\mathrm{Zim}_h^n(\mathcal{Z})})$ respectively. Then $\mathcal{V}_n = \mathrm{Im}_h^n(\mathcal{U})$.

Proof: By induction. When $n = 1$, this is R32.4.10. If true for n, by R32.4.10 again $\mathcal{V}_{n+1} = \mathrm{Im}_h(\mathcal{V}_n) = \mathrm{Im}_h(\mathrm{Im}_h^n(\mathcal{U}) = \mathrm{Im}_h^{n+1}(\mathcal{U})$.

Proposition R32.4.15 Let $n \in \mathbb{N}$. Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}$ be a normal basis for (X, τ). Let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Assume, for every $Z \in \mathcal{Z}$, $h^{-1}[Z]$ is in $\mathcal{Z}$. Let $\mathcal{V}_n, \mathcal{V}_{n+1}$ be the separated totally bounded uniformities corresponding to the compactification classes of $(\omega(\mathrm{Zim}_h^n(\mathcal{Z})), \iota_{\mathrm{Zim}_h^n(\mathcal{Z})})$ and $(\omega(\mathrm{Zim}_h^{n+1}(\mathcal{Z})), \iota_{\mathrm{Zim}_h^{n+1}(\mathcal{Z})})$ respectively. Then $h : (X, \mathcal{V}_n) \to (X, \mathcal{V}_{n+1})$ is a unimorphism.

Proof: Let $\mathcal{U}$ be the separated totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. By R34.4.1 $h : (X, \mathcal{U}) \to (X, \mathcal{U})$ is uniformly continuous. The result now follows from R32.2.9iv and R32.4.14.

Lemma R32.4.16 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{Z}$ be a normal basis for (X, τ). Assume $h^{-1}[Z] \in \mathcal{Z}$ for every $Z \in \mathcal{Z}$. Then

i) $\mathcal{Z} \subseteq \mathrm{Zim}_h(\mathcal{Z})$.

ii) $\mathrm{Zim}_h^n(\mathcal{Z}) \subseteq \mathrm{Zim}_h^{n+1}(\mathcal{Z})$ for every $n \in \mathbb{N}$.

iii) $\cup_{n=1}^{\infty}\mathrm{Zim}_h^n(\mathcal{Z})$ is a normal basis for (X, τ).

Proof: For i) let $Z \in \mathcal{Z}$. By hypothesis $h^{-1}[Z]$ is in $\mathcal{Z}$ and by definition $Z = h[h^{-1}[Z]] \in \mathrm{Zim}_h(\mathcal{Z})$. For ii), by R32.4.13ii a typical element of $\mathrm{Zim}_h^n(\mathcal{Z})$ is $h^n[Z]$ for some $Z \in \mathcal{Z}$. Again $h^{-1}[Z]$ is in $\mathcal{Z}$ and so $h^{n+1}[h^{-1}[Z]] = h^n[Z]$ is in $\mathrm{Zim}_h^n(\mathcal{Z})$. Part iii) follows from ii) and R9.2.1.

Definition R32.4.17 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{Z}$ be a normal basis for (X, τ). Assume $h^{-1}[Z] \in \mathcal{Z}$ for every $Z \in \mathcal{Z}$. $\mathcal{Z}_h(\mathcal{Z})$ is defined to be $\cup_{n=1}^{\infty}\mathrm{Zim}_h^n(\mathcal{Z})$.

Proposition R32.4.18 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{Z}$ be a normal basis for (X, τ) with $h^{-1}[Z] \in \mathcal{Z}$ for every $Z \in \mathcal{Z}$. Let $\mathcal{U}$ be the separated totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. Then $\mathcal{V}_h(\mathcal{U})$ is the separated totally bounded uniformity corresponding to the class of $(\omega(\mathcal{Z}_h(\mathcal{Z})), \iota_{\mathcal{Z}_h(\mathcal{Z})})$.

Proof: Let V be in the uniformity corresponding to the class of $(\omega(\mathcal{Z}_h(\mathcal{Z})), \iota_{\mathcal{Z}_h(\mathcal{Z})})$. By R14.1.3 There are $Z_1, \ldots, Z_j$ in $\mathcal{Z}_h(\mathcal{Z})$ such that $\cup_{i=1}^{j}(X - Z_i) \times (X - Z_i) \subseteq V$ and $\cap_{i=1}^{j} Z_i = \emptyset$. Because the sequence of normal bases is increasing (R32.4.16ii), there is m such that $Z_1, \ldots, Z_j$ are all in $\mathrm{Zim}_h^m(\mathcal{Z})$ and so by R32.4.14 and R14.1.3 again V is in $\mathrm{Im}_h^m(\mathcal{U})$, which is contained in $\mathcal{V}_h(\mathcal{U})$ by R32.2.15i. Conversely let W be in $\mathcal{V}_h(\mathcal{U})$. By R32.2.15i W is in $\mathrm{Im}_h^n(\mathcal{U})$ for some n. Because R32.4.14 holds and $\mathrm{Zim}_h^n(\mathcal{Z}) \subseteq \mathcal{Z}_h(\mathcal{Z})$, by R14.1.3 W is in the uniformity corresponding to the class of $(\omega(\mathcal{Z}_h(\mathcal{Z})), \iota_{\mathcal{Z}_h(\mathcal{Z})})$.

Corollary R32.4.19 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{Z}$ be a normal basis for (X, τ) with $h^{-1}[Z] \in \mathcal{Z}$ for every $Z \in \mathcal{Z}$. Then h extends to an auto-homeomorphism of $\omega(\mathcal{Z}_h(\mathcal{Z}))$.

Proof: Let $\mathcal{U}$ be the separated totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. By R32.4.1 $h : (X, \mathcal{U}) \to (X, \mathcal{U})$ is uniformly continuous. By R32.2.17 h extends to an auto-homeomorphism of any compactification in the class corresponding to $\mathcal{V}_h(\mathcal{U})$ and so by R32.4.18 the conclusion holds.

Proposition R32.4.20 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{Z}$ be a normal basis for (X, τ) with $h[Z]$ and $h^{-1}[Z]$ in $\mathcal{Z}$ for every $Z \in \mathcal{Z}$. Then $\mathrm{Zim}_h^n(\mathcal{Z}) = \mathcal{Z}$ for every $n \in \mathbb{N}$ and $\mathcal{Z}_h(\mathcal{Z}) = \mathcal{Z}$.

Proof: By induction. By definition of $\mathrm{Zim}_h(\mathcal{Z}) = \mathrm{Zim}_h^1(\mathcal{Z})$, $h[Z] \in \mathcal{Z}$ for every $Z \in \mathcal{Z}$ implies $\mathrm{Zim}_h^1(\mathcal{Z}) \subseteq \mathcal{Z}$. R32.4.16 shows the opposite containment. If $\mathrm{Zim}_h^n(\mathcal{Z}) = \mathcal{Z}$, the first case shows $\mathrm{Zim}_h^{n+1}(\mathcal{Z}) = \mathcal{Z}$. The second claim now follows from R32.4.17.

By R32.4.3 the hypotheses of R32.4.20 imply h extends to an auto-homeomorphism of $\omega(\mathcal{Z})$. The weaker hypothesis of R32.4.19 implies only that h extends continuously to $\omega(\mathcal{Z})$. The following corollary is an application of R32.2.12 to the context of Wallman compactifications.

Corollary R32.4.21 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $h : (X, \tau) \to (X, \tau)$ be an auto-homeomorphism. Let $\mathcal{Z}$ be a normal basis for (X, τ) with $h^{-1}[Z] \in \mathcal{Z}$ for every $Z \in \mathcal{Z}$. Assume h does not extend to an auto-homeomorphism of $\omega(\mathcal{Z})$. Then h does not extend to an auto-homeomorphism of $\omega(\mathrm{Zim}_h^n(\mathcal{Z}))$ for every $n \in \mathbb{N}$.

Proof: Let $\mathcal{U}$ be the separated totally bounded uniformity corresponding to the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. R32.4.1 and R7.1.3 imply h extends continuously to $\omega(\mathcal{Z})$. By R32.4.14 $(\omega(\mathrm{Zim}_h^n(\mathcal{Z})), \iota_{\mathrm{Zim}_h^n(\mathcal{Z})})$ is in the compactification class corresponding to $\mathrm{Im}_h^n(\mathcal{U})$. The conclusion is now immediate from R32.2.12.

Results for $\mathbb{N}_k$ and $\mathbf{R}_k$

Notation, definitions, and results from [7] and [4] will be reviewed first. For $n, k, i \in \mathbb{N}$ with $k \geq 2$, $C_n^i(k)$ denotes the equivalence class of i mod k^n. For $S \subseteq \mathbb{N}$ and $\Delta \subseteq \{1, \ldots k^n\}$, S is associated with Δ provided $j \in \Delta$ implies $S \cap C_n^j(k)$ is finite and $j \notin \Delta$ implies $(X - S) \cap C_n^j(k)$ is finite. $\mathcal{Z}(n,k)$, which is defined as the collection of all subsets of $\mathbb{N}$ associated with some $\Delta \subseteq \{1, \ldots, k^n\}$, is a normal basis for $\mathbb{N}$ with the discrete topology, as is $\mathcal{Z}_k = \cup_{n=1}^{\infty} \mathcal{Z}(n,k)$. $\mathcal{Z}(n,k)$ is closed under complementation for all n and so $\mathcal{Z}_k$ is also. $\mathbb{N}_k$ denotes the compactification $\omega(\mathcal{Z}_k)$ with the embedding that takes a positive integer to its point-filter. Because the underlying topological space is discrete, its auto-homeomorphisms are simply the permutations.

Lemma R32.5.1 Let $k \in \mathbb{N}$ with $k \geq 2$ and let σ be a permutation of $\mathbb{N}$. Assume that, for some n, m in $\mathbb{N}$, $\sigma[C_n^i(k)] \in \mathcal{Z}(m,k)$ for every i in $\{1, \ldots, k^n\}$. Let $Z \in \mathcal{Z}(n,k)$. Then $\sigma[Z] \in \mathcal{Z}(m,k)$.

Proof: Finite subsets of $\mathbb{N}$ are in $\mathcal{Z}(m,k)$, being associated with $\{1, \ldots, k^m\}$. By definition $Z \in \mathcal{Z}(n,k)$ implies that, for every $i \in \{1, \ldots, k^n\}$, either $A_i = Z \cap C_n^i(k)$ is finite or $B_i = (\mathbb{N} - Z) \cap C_n^i(k)$ is finite. Note that $C_n^i(k) = A_i \cup B_i$ and $A_i \cap B_i = \emptyset$ so that $A_i = C_n^i(k) \cap (\mathbb{N} - B_i)$. It will be shown that $\sigma[A_i]$ is in $\mathcal{Z}(m,k)$ for all i. When A_i is finite, $\sigma[A_i]$ is in $\mathcal{Z}(m,k)$. When A_i is not finite, $\sigma[B_i] \in \mathcal{Z}(m,k)$, as is its complement. By hypothesis $\sigma[C_n^i(k)]$ is in $\mathcal{Z}(m,k)$. Now $\sigma[A_i] = \sigma[C_n^i(k)] \cap (\mathbb{N} - \sigma[B_i])$ because σ is a permutation. Since $\mathcal{Z}(m,k)$ is closed under finite intersections, $\sigma[A_i]$, is also in $Z(m,k)$. Since $Z = \cup_{i=1}^{k^n} A_i$ and a normal basis is closed under finite unions, $\sigma[Z] = \cup_{i=1}^{k^n} \sigma[A_i]$ is in $\mathcal{Z}(m,k)$.

Corollary R32.5.2 Let $k \in \mathbb{N}$ with $k \geq 2$ and let σ be a permutation of $\mathbb{N}$. Then $\sigma[Z] \in \mathcal{Z}_k$ for every $Z \in \mathcal{Z}_k$ if and only if $\sigma[C_n^j(k)] \in \mathcal{Z}_k$ for every $n \in \mathbb{N}$ and j in $\{1, \ldots, k^n\}$.

Proof: The condition is necessary because each $C_n^j(k)$ is in $\mathcal{Z}(n,k)$, being associated with $\{1, \ldots, k^n\} - \{j\}$. Now assume $\sigma[C_n^j(k)] \in \mathcal{Z}_k$ for every $n \in \mathbb{N}$ and j in $\{1, \ldots, k^n\}$ and let $Z \in \mathcal{Z}_k$. Pick $n \in \mathbb{N}$ with $Z \in \mathcal{Z}(n,k)$. $\mathcal{Z}(t,k) \subseteq \mathcal{Z}(t+1,k)$ for all $t \in \mathbb{N}$ and so there is $m \in \mathbb{N}$ such that $\sigma[C_n^j] \in \mathcal{Z}(m,k)$ for every $j \in \{1, \ldots, k^n\}$. By the previous lemma, $\sigma[Z] \in \mathcal{Z}(m,k) \subseteq \mathcal{Z}_k$.

Corollary R32.5.3 Let $k \in \mathbb{N}$ with $k \geq 2$ and let σ be a permutation of $\mathbb{N}$. Then σ extends to an auto-homeomorphism of $\mathbb{N}_k$ if and only if $\sigma[C_n^j(k)]$ and $\sigma^{-1}[C_n^j(k)]$ are in $\mathcal{Z}_k$ for every $n \in \mathbb{N}$ and j in $\{1, \ldots, k^n\}$.

Proof: If σ extends to an auto-homeomorphism of $\mathbb{N}_k$, the condition follows from R32.4.7 and R32.5.2. The converse follows from R32.5.2 applied to σ^{-1} as well as σ and R32.4.3.

The next result uses the notion of order of an element in the group of permutations with composition as operation.

Corollary R32.5.4 Let $k \in \mathbb{N}$ with $k \geq 2$ and let σ be a permutation of $\mathbb{N}$ of finite order. Then σ extends to an auto-homeomorphism of $\mathbb{N}_k$ if and only if $\sigma[C_n^j(k)] \in \mathcal{Z}_k$ for every $n \in \mathbb{N}$ and j in $\{1, \ldots, k^n\}$.

Proof: The condition is necessary for any permutation by by R32.5.3. For sufficiency, if σ has order 1, σ is the identity, which extends to the identity. If σ has order $m \geq 2$, $\sigma^{-1} = \sigma^{m-1}$. A routine induction shows that $\sigma^{m-1}[C_n^j(k)] \in \mathcal{Z}_k$ for every $n \in \mathbb{N}$ and j in $\{1, \ldots, k^n\}$ and so σ extends by the previous corollary.

On this site $\mathbf{R}_k$ with $k \geq 2$ has usually denoted the compactification $(\mathbf{R}_k, f_k)$ of $(\mathbf{Z}, \tau_k)$, where $\mathbf{R}_k$ is the remnant space obtained by removing the point-filters (images of elements in $\mathbb{N}$) from $\mathbb{N}_k$ and $f_k(z)$ is the non-point filter associated with $\{x_n\}$, where $x_n \equiv z \bmod k^n$ and $x_n \in \{1, 2, \ldots, k^n\}$. With $D_n^z(k)$ defined as the equivalence class of $z \bmod k^n$ in $\mathbf{Z}$, $\mathcal{B}_k = \{D_n^z(k) : n \in \mathbb{N} \text{ and } z \in \mathbf{Z}\}$ is a clopen basis for τ_k. In [10] $\mathcal{D}_k$, the set of all unions of finite subcollectios from $\mathcal{B}_k$, is shown to be a normal basis for $(\mathbf{Z}, \tau_k)$. $\mathcal{D}_k$ is closed under complementation and the associated Wallman compactification $(\omega(\mathcal{D}_k), \delta_k)$ is equivalent to $(\mathbf{R}_k, f_k)$ (R27.1.10).

Lemma R32.5.5 Let $k \in \mathbb{N}$ with $k \geq 2$ and let $h : \mathbf{Z} \to \mathbf{Z}$. Then $h[Z] \in \mathcal{D}_k$ for every $Z \in \mathcal{D}_k$ if and only if $h[D_n^z(k)] \in \mathcal{D}_k$ for every $n \in \mathbb{N}$ and $z \in \mathbf{Z}$.

Proof: The condition is necessary because $\mathcal{B}_k \subseteq \mathcal{D}_k$. Conversely, assume $h[D_n^z(k)]$ is in $\mathcal{D}_k$ for every $n \in \mathbb{N}$ and $z \in \mathbf{Z}$, i.e., $h[B] \in \mathcal{D}_k$ for every $B \in \mathcal{B}_k$. Let $Z \in \mathcal{D}_k$. By definition of $\mathcal{D}_k$, Z is a finite union of elements of $\mathcal{B}_k$ and so $h[Z]$ is a finite union of elements of the normal basis $\mathcal{D}_k$, which is closed under finite unions.

Proposition R32.5.6 Let $k \in \mathbb{N}$ with $k \geq 2$ and let h be a permutation of $\mathbf{Z}$. Then h extends (relative to f_k) to an auto-homeomorphism of $\mathbf{R}_k$ if and only if $h[D_n^z(k)]$ and $h^{-1}[D_n^z(k)]$ are in $\mathcal{D}_k$ for every $n \in \mathbb{N}$ and $z \in \mathbf{Z}$.

Proof: By R32.4.3 and R32.4.7 h extends to an auto-homeomorphism of $\omega(\mathcal{D}_k)$ if and only if $h[Z]$ and $h^{-1}[Z]$ are in $\mathcal{D}_k$ for every $Z \in \mathcal{D}_k$. By R32.1.3, since $(\omega(\mathcal{D}_k), \delta_k)$ is equivalent to $(\mathbf{R}_k, f_k)$, $\omega(\mathcal{D}_k)$ can be replaced in that statement with $\mathbf{R}_k$. The conclusion follows by applying applying the previous lemma to both h and h^{-1}.

R27.4.4 shows that $(\mathbf{R}_k, g_k)$ is a T_2 compactification of $(\mathbb{N}, \sigma_k)$, where g_k is f_k restricted to $\mathbb{N}$ and σ_k is the relative topology induced on $\mathbb{N}$ from τ_k. The collection $\{C_n^j(k) : n, j \in \mathbb{N}\}$ is a clopen basis for σ_k.

This compactification can be represented as a Wallman compactification as follows. Let $\mathcal{C}_k$ be $\{D \cap \mathbb{N} : D \in \mathcal{D}_k\}$. (Note that for $n, j \in \mathbb{N}$, $D_n^j(k) \cap \mathbb{N} = C_n^j(k)$.) R27.4.14 shows that $\mathcal{C}_k$ is a normal basis for $(\mathbb{N}, \sigma_k)$ and the corresponding Wallman compactification $(\omega(\mathcal{C}_k), \epsilon_k)$ is equivalent to $(\mathbf{R}_k, g_k)$. $\mathcal{C}_k$ is also closed under complementation.

Lemma R32.5.7 Let $k \in \mathbb{N}$ with $k \geq 2$ and let $h : \mathbb{N} \to \mathbb{N}$. Then $h[Z] \in \mathcal{C}_k$ for every $Z \in \mathcal{C}_k$ if and only if $h[C_n^j(k)] \in \mathcal{C}_k$ for every $n, j \in \mathbb{N}$.

Proof: It is noted in [10] that $\mathcal{C}_k$ is the set of unions of finite collections of $\{C_n^j(k) : n, j \in \mathbb{N}\}$. The argument follows the same pattern as the proof of R32.5.5.

Proposition R32.5.8 Let $k \in \mathbb{N}$ with $k \geq 2$ and let h be a permutation of $\mathbb{N}$. Then h extends (relative to g_k) to an auto-homeomorphism of $\mathbf{R}_k$ if and only if $h[C_n^j(k)]$ and $h^{-1}[C_n^j(k)]$ are in $\mathcal{C}_k$ for every $n, j \in \mathbb{N}$.

Proof: Similar to the proof of R32.5.6.

By R27.Add.3 $(\mathbf{R}_k, g_k) \leq \mathbb{N}_k$ under the generalized ordering of compactifications described in [8]. The following corollary points out a case in which a homeomorphism which extends to an auto-homeomorphism of the smaller compactification must also extend for the larger, a relationship which does not hold in general as noted in the comment at the end of R32.3.

Corollary R32.5.9 Let $k \in \mathbb{N}$ with $k \geq 2$ and let h be a permutation of $\mathbb{N}$. Assume h extends (relative to g_k) to an auto-homeomorphism of $\mathbf{R}_k$. Then h extends to an auto-homeomorphism of $\mathbb{N}_k$.

Proof: This follows from R32.5.8 and R32.5.3 because $\mathcal{C}_k \subseteq \mathcal{Z}_k$.

The following example shows that the converse of R32.5.9 is false.

Example R32.5.10 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $h : \mathbb{N} \to \mathbb{N}$ be defined by $h(1) = 2, h(2) = 1$ and $h(n) = n$ for $n \geq 3$. Clearly h is a permutation of $\mathbb{N}$. For $n \in \mathbb{N}$ and $j \in \{1, \ldots, k^n\}$, $h[C_n^j(k)] = (C_n^j(k) - \{1,2\}) \cup S$, where $S \subseteq \{1,2\}$. Then $h[C_n^j(k)]$ is associated with $\{1, \ldots, k^n\} - \{j\}$ and so is in $\mathcal{Z}(n,k) \subseteq \mathcal{Z}_k$. Because h has order 2, by R32.5.4 h extends to an auto-homeomorphism of $\mathbb{N}_k$. Next note that if $2 \in C_n^j(k)$, then infinitely many even integers must be in $C_n^j(k)$, because $2 + 2mk^n \equiv 2 \mod k^n$ for all m in $\mathbb{N}$. In particular, with $k = 2$, $h[C_1^1(2)] = \{2\} \cup (C_1^1(2) - \{1\})$ contains no even number except 2. Since an element of $\mathcal{C}_2$ must be a finite union of classes from $\{C_n^j(2) : n, j \in \mathbb{N}\}$, $h[C_1^1(2)] \notin \mathcal{C}_2$. By R32.5.8 h does not extend to an automorphism of $\mathbf{R}_2$ relative to g_k.

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References

1. This Website, P2: Uniform Spaces
2. This Website, P3: Normal Bases
3. This Website, R1: Existence of Suprema via Uniform Space Theory
4. This Website, R5: Finite-point Compactifications
5. This Website, R7: Uniform Continuity and Extension of Maps
6. This Website, R9: Directed Sets of Normal Bases
7. This Website, R10: Some Metric Compactifications of the Natural Numbers
8. This Website, R13: Mixed Suprema
9. This Website, R14: Uniformities and Normal Bases
10. This Website, R27: Normal Bases for the Remnant Rings

Added 2023

This addition fills a gap in the above by showing there is an example of a compactification and an auto-homeomorphism of the underlying $T_{3\frac{1}{2}}$ space that extends to a continuous map which is not a homeomorphism of the compactification.

By R32.1.2iii and R7.1.3 the existence of such an auto-homeomorphism is equivalent to the existence of a separated, totally bounded uniform space $(X, \mathcal{V})$ and a uniformly continuous map $f : (X, \mathcal{V}) \to (X, \mathcal{V})$ such that $f : (X, \tau(\mathcal{V})) \to (X, \tau(\mathcal{V})$ is a homeomorphism but f is not a unimorphism from $(X, \mathcal{V})$ to $(X, \mathcal{V})$, i.e., f^{-1} is not uniformly continuous from $(X, \mathcal{V})$ to $(X, \mathcal{V})$.

Throughout this added subsection X will denote the interval of real numbers $(1, \infty)$ and $\mathcal{U}$ the uniformity on X generated by the absolute value metric, i.e., $\mathcal{U}$ has basis

$\{V_\epsilon : \epsilon > 0\}$ where $V_\epsilon = \{(x,y) \in X \times X : |x-y| < \epsilon\}$. The maps $f : X \to X$ and $g : X \to X$ will be defined by $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Lemma R32.Add.1 The following hold.

i) f and g are bijections and $g = f^{-1}$.

ii) $f : (X, \tau(\mathcal{U})) \to (X, \tau(\mathcal{U}))$ is a homeomorphism.

iii) $f : (X,\mathcal{U}) \to (X,\mathcal{U})$ is uniformly continuous.

iv) g is not uniformly continuous from $(X,\mathcal{U})$ to $(X,\mathcal{U})$.

Proof: It is easy to check that $f \circ g = g \circ f = \mathrm{id}_X$ and so i) holds. From calculus both f and g are continuous. Since $\tau(\mathcal{U})$ is the usual topology on X, ii) holds. By the Mean Value Theorem for, $x, y \in X$, $|f(x) - f(y)| \leq \frac{1}{2}|x-y|$ because $|f'(t)| \leq \frac{1}{2}$ for all t in X. The inequality implies $\epsilon - \delta$ uniform continuity, which is equivalent to uniform continuity from $(X,\mathcal{U})$ to $(X,\mathcal{U})$. Thus iii) holds. Lastly let $\epsilon = 1$ and let $\delta > 0$. Pick $n \in \mathbb{N}$ such that $n > 1/\delta$. Then $(n+\delta/2)^2 - n^2 > n\delta > 1$ but $(n+\delta/2) - n < \delta$ and so g is not uniformly continuous.

Of course, $(X,\mathcal{U})$ is not totally bounded and so the lemma does not provide the desired example. However the uniformiy generated by $\mathcal{U}$-proximal covers does.

Recall from R8.Add.3 that a $\mathcal{U}$-proximal cover of X is a finite collection $\{A_1, \ldots, A_n\}$ of subsets of X for which there exist sets $B_1, \ldots, B_n$ and $U \in \mathcal{U}$ such that $X = \cup_{i=1}^n B_i$ and $U[B_i] \subseteq A_i$ for $1 \leq i \leq n$. The uniformity generated by $\mathcal{U}$-proximal covers is $\mathcal{V}$, defined as the set $\{V \subseteq X \times X :$ there is a proximal cover $\{A_1, \ldots, A_n\}$ with $\cup_{i=1}^n A_i \times A_i \subseteq V\}$. By R8.Add.5 $\mathcal{V}$ is a totally bounded uniformity contained in $\mathcal{U}$ such that $\tau(\mathcal{V}) = \tau(\mathcal{U})$.

The next lemma is known but is included here for convenience of reference.

Lemma R32.Add.2 Let $(A,\mathcal{U}_1)$ and $(B,\mathcal{U}_2)$ be uniform spaces and assume $h : (A,\mathcal{U}_1) \to (B,\mathcal{U}_2)$ is uniformly continuous. Let $\mathcal{V}_i$ be the uniformity generated by $\mathcal{U}_i$-proximal covers. Then $h : (A, \mathcal{V}_1) \to (B, \mathcal{V}_2)$ is uniformly continuous.

Proof: Let $V \in \mathcal{V}_2$. There is a finite collection $\{A_1, \ldots, A_n\}$ of subsets of B, sets $B_1, \ldots, B_n$, and $U \in \mathcal{U}_2$ with $B = \cup_{i=1}^n B_i$ and $U[B_i] \subseteq A_i$ for $1 \leq i \leq n$ such that $\cup_{i=1}^n A_i \times A_i \subseteq V$. Let $W = (h \times h)^{-1}[V]$. By the hypothesis of uniform continuity and the fact that $\mathcal{V}_2 \subseteq \mathcal{U}_2$, $W \in \mathcal{U}_1$. It is easy to check that $A = \cup_{i=1}^n h^{-1}[B_i]$, $\cup_{i=1}^n h_i^{-1}[A_i] \times h^{-1}[A_i] \subseteq (h \times h)^{-1}[V]$, and, for $\leq i \leq n$, $W[h^{-1}[B_i]] \subseteq h^{-1}[A_i]$. It follows that $\{h^{-1}[A_1], \ldots, h^{-1}[A_n]\}$ is a $\mathcal{U}_1$-proximal cover of A. Thus $(h \times h)^{-1}[V] = W$ is in $\mathcal{V}_1$ as required for the conclusion.

Corollary R32.Add.3 $f : (X, \mathcal{V}) \to (X, \mathcal{V})$ is uniformly continuous.

Proof: This follows from R32.Add.1iii and the lemma.

To produce the desired example, it still needs to be verified that g is not uniformly continuous from $(X, \mathcal{V})$ to $(X, \mathcal{V})$.

First an element of $\mathcal{V}$ will be identified. Let $B_1 = (1,2] \cup (\cup_{n=1}^\infty [2n+1, 2n+2])$ and $B_2 = (1,\infty) - B_1 = \cup_{n=1}^\infty (2n, 2n+1)$. By definition $\{V_{0.1}[B_1], V_{0.1}[B_2]\}$ is a $\mathcal{U}$-proximal cover of X and so $W = (V_{0.1}[B_1] \times V_{0.1}[B_1]) \cup (V_{0.1}[B_2] \times V_{0.1}[B_2])$ is in $\mathcal{V}$.

Lemma R32.Add.4 Let $j \in \mathbb{N}$ and $x, y \in X$.

i) If $2j + 0.1 < x < 2j + 0.9$, then $x \in V_{0.1}[B_2]$ and $x \notin V_{0.1}[B_1]$.

ii) If $2j + 1.1 < y < 2j + 1.9$, then $y \in V_{0.1}[B_1]$ and $y \notin V_{0.1}[B_2]$.

Proof: It is easy to check that $V_{0.1}[B_1] = (1, 2.1) \cup (\cup_{n=1}^\infty (2n+0.9, 2n+2.1))$ and $V_{0.1}[B_2] = \cup_{n=1}^\infty (2n-0.1, 2n+1.1)$. Let $2j+0.1 < x < 2j+0.9$ so that clearly $x \in V_{0.1}[B_2]$.

Since $j \geq 1$, $2.1 < x$ so that $x \notin (1, 2.1)$. For $n \leq j-1$, $2n+2.1 \leq 2(j-1)+2.1 = 2j+0.1 < x$ so that $x \notin (2n+0.9, 2n+2.1)$. For $n \geq j$, $x < 2j+0.9 \leq 2n+0.9$ and so $x \notin (2n+0.9, 2n+2.1)$. Thus $x \notin V_{0.1}[B_1]$ and i) holds. Now assume $2j+1.1 < y < 2j+1.9$. Then $y \in (2j+0.9, 2j+2.1) \subseteq V_{0.1}[B_1]$. Next let $n \in \mathbb{N}$. If $n \leq j$, $2n+1.1 \leq 2j+1.1 < y$ and so $y \notin (2n-0.1, 2n+1.1)$. If $n > j$, $y < 2j+1.9 = 2(j+1)-0.1 \leq 2n-0.1$ so that $y \notin (2n-0.1, 2n+1.1)$. Thus $y \notin V_{0.1}[B_2]$ and ii) holds.

Lemma R32.Add.5 g is not uniformly continuous from $(X, \mathcal{V})$ to $(X, \mathcal{V})$.

Proof: It will be shown that $(g \times g)^{-1}[W]$ is not in $\mathcal{U}$. This is sufficient because $\mathcal{V} \subseteq \mathcal{U}$ by R8.Add.5iii. Let $\delta > 0$. Since $\lim_{n\to\infty} \frac{1}{\sqrt{2n+0.5}+\sqrt{2n+1.5}} = 0$, there is $N \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{2N+0.5}+\sqrt{2N+1.5}} < \delta.$$

Let $s = \sqrt{2N+0.5}$ and $t = \sqrt{2N+1.5}$. Then $t^2 - s^2 = 1$ and $t - s = \frac{1}{t+s} < \delta$ so that $(t, s) \in V_\delta$. Also $2N+0.1 < s^2 < 2N+0.9$ so that $s^2 \notin V_{0.1}[B_1]$ by R32.Add.4i. Similarly, $2N+1.1 < t^2 < 2N+1.9$ so that $t^2 \notin V_{0.1}[B_2]$ by R32.Add.4ii. By the definition of W, $(g \times g)(t, s) = (t^2, s^2) \notin W$, i.e., $V_\delta \not\subseteq (g \times g)^{-1}[W]$. Thus the claim is verified and the conclusion follows.

The next two corollaries merely summarize the example.

Corollary R32.Add.6 Let $X = (1, \infty)$ have the usual topology and let $\mathcal{U}$ be the uniformity for X generated by the absolute value metric. Let $\mathcal{V}$ be the uniformity for X generated by the $\mathcal{U}$-proximal covers and let $f : X \to X$ by $f(x) = \sqrt{x}$. Then

i) $f : (X, \tau(\mathcal{V})) \to (X, \tau(\mathcal{V}))$ is a homeomorphism.

ii) $f : (X, \mathcal{V}) \to (X, \mathcal{V})$ is uniformly continuous.

iii) f is not a unimorphism from $(X, \mathcal{V})$ to $(X, \mathcal{V})$.

Proof: Part i) follows from R32.Add.1ii and R8.Add.5iv. Part ii) repeats R32.Add.3. Part iii) follows from R32.Add.1i and R32.Add.5.

Corollary R32.Add.7 Let $X = (1, \infty)$ have the usual topology and let $\mathcal{U}$ be the uniformity for X generated by the absolute value metric. Let $\mathcal{V}$ be the uniformity for X generated by the $\mathcal{U}$-proximal covers and let (Y, h) be in the compactification class corresponding to $\mathcal{V}$. Let $f : X \to X$ by $f(x) = \sqrt{x}$. Then f is an auto-homeomorphism of $(X, \tau(\mathcal{V}))$ which extends continuously to Y but the extension is not a homeomorphism.

Proof: The first assertion follows from the definition of auto-homeomorphism and R32.Add.6i. By R32.Add.6ii and R7.1.3 f extends continuously to Y. The extension is not a homeomorphism by R32.Add.6iii and R32.1.2.

The example also allows the following simple observations: Let (Y, h) be a T_2 compactification in the class corresponding to $\mathcal{V}$. By R32.1.4 (Y, h) is not the Stone-Čech compactification and by R32.3.1 it is not a finite point compactification.

Finally, the example provides additional instances of non-equivalent compactifications with homeomorphic compact spaces. The result will be presented generally because it applies to any map with the properties of f. The first two lemmas may be considered obvious but haven't been explicitly recorded here.

Lemma R32.Add.8 Let $(S, \mathcal{S})$ and $(R, \mathcal{R})$ be separated, totally bounded uniform spaces. Let (A, α) and (B, β) be in the compactification classes corresponding to $\mathcal{S}$, $\mathcal{R}$

respectively. Let $\psi : S \to R$. Assume $\Psi : A \to B$ is continuous with $\Psi \circ \alpha = \beta \circ \psi$. Then $\psi : (S, \mathcal{S}) \to (R, \mathcal{R})$ is uniformly continuous.

Proof: By R1.6a α and β are uniform embeddings of $(S, \mathcal{S})$ and $(R, \mathcal{R})$ respectively, i.e., they are unimorphisms onto their images with the subspace uniformities from the unique uniformities for A, B respectively. Since A is compact, Ψ is uniformly continuous. By hypothesis $\psi = \beta^{-1} \circ \Psi|_{\alpha[X]} \circ \alpha$. As the composition of uniformly continuous maps, ψ is uniformly continuous.

Lemma R32.Add.9 Let $(S, \mathcal{S})$ and $(R, \mathcal{R})$ be separated, totally bounded uniform spaces. Let (A, α) and (B, β) be in the compactification classes corresponding to $\mathcal{S}$, $\mathcal{R}$ respectively. Let $\psi : S \to R$. If $\psi : (S, \mathcal{S}) \to (R, \mathcal{R})$ is a unimorphism, then ψ extends to a homeomorphism from A onto B. If ψ is onto and ψ extends to a homeomorphism from A onto B, then $\psi : (S, \mathcal{S}) \to (R, \mathcal{R})$ is a unimorphism.

Proof: First assume ψ is a unimorphim. By R7.1.3 ψ and ψ^{-1} extend to continuous maps F and G, i.e., $F \circ \alpha = \beta \circ \psi$ and $G \circ \beta = \alpha \circ \psi^{-1}$. It follows that $G \circ F \circ \alpha = \alpha$, i.e., the continuous $G \circ F$ agrees with id_A, the identity map on A, on the dense subset $\alpha[S]$. Since A is T_2, $G \circ F = \mathrm{id}_A$. Similarly, $F \circ G = \mathrm{id}_B$ and so $G = F^{-1}$, i.e., F is a homeomorphism onto B. Now assume ψ is onto and extends to a homeomorphism F from A to B. By R32.Add.8 $\psi : (S, \mathcal{S}) \to (R, \mathcal{R})$ is a uniformly continuous. Because $F \circ \alpha = \beta \circ \psi$ and F and α are one-to-one, ψ must be one-to-one. Since ψ is onto by hypothesis, it follows that $\alpha \circ \psi^{-1} = F^{-1} \circ \beta$, i.e., F^{-1} is a continuous extension of ψ^{-1}. By R32.Add.8 again ψ^{-1} is also uniformly continuous, i.e., $\psi : (S, \mathcal{S}) \to (R, \mathcal{R})$ is a unimorphism.

Comment: Without the assumption that ψ is onto, the second part of the lemma fails. Let $(R, \mathcal{R})$ be separated, totally bounded uniform space. Assume S is a $\tau(\mathcal{R})$-dense proper subset of R and let $\mathcal{S}$ be the subspace uniformity from $\mathcal{R}$ on S. Let $\psi : S \to R$ be the inclusion map, which is not onto so that ψ cannot be a unimorphism. Now let (B, β) be in the compactification class corresponding to $\mathcal{R}$. By R27.4.1 $(B, \beta|_S)$ is in the compactification class corresponding to $\mathcal{S}$. ψ extends to the identity map on B, a homeomorphism.

Proposition R32.Add.10 Let $(Z, \mathcal{W})$ be a separated, totally bounded uniform space and let $\sigma : (Z, \tau(\mathcal{W})) \to (Z, \tau(\mathcal{W}))$ be an auto-homeomorphism. Assume σ is uniformly continuous from $(Z, \mathcal{W})$ to $(Z, \mathcal{W})$ but not a unimorphism. Let (Y, h) be in the compactification class corresponding to $\mathcal{W}$ and (Y_1, h_1) be in the compactification class corresponding to $\mathrm{Im}_\sigma(\mathcal{W})$. Then

i) $(Y, h) \leq (Y_1, h_1)$ but (Y, h) is not equivalent to (Y_1, h_1).

ii) Y is homeomorphic to Y_1.

Proof: By R32.2.3 $\mathcal{W} \subseteq \mathrm{Im}_\sigma(\mathcal{W})$. By R32.2.4 and R32.1.2 $\mathcal{W} \neq \mathrm{Im}_\sigma(\mathcal{W})$. Part i) now follows from R1.5. By R32.2.2iv $\sigma : (Z, \mathcal{W}) \to (Z, \mathrm{Im}_\sigma(\mathcal{W}))$ is a unimorphism and so by R32.Add.9 σ extends to a homeomorphism from Y to Y_1. Thus part ii) holds.

Continue with the hypotheses of R32.Add.10. Let (Y_n, h_n) be in the compactification class corresponding to $\mathrm{Im}^n_\sigma(\mathcal{W})$. By using R32.2.9, R32.2.11, and similar arguments, it can be shown that $(Y_n, h_n) \leq (Y_{n+1}, h_{n+1})$, (Y_n, h_n) is not equivalent to (Y_{n+1}, h_{n+1}), and Y_n is homeomorphic to Y_{n+1} for every n. R32.2.9v and R32.Add.9 (or a routine induction) show that Y is homeomorphic to Y_n for every n.

Added Reference

11. This Website, R8: Lattice and Semilattice Properties

Added 2024

This added subsubsection notes that analogs of various results can be derived for $\mathbb{N}_\infty$, $(\mathbf{R}_\infty, f_\infty)$, and $(\mathbf{R}_\infty, g_\infty)$. The notation to be used was described in R32.5.

The first lemma is slight extension of R32.5.1.

Lemma R32.Add.11 Let $k, j \in \mathbb{N}$ with both k and j greater than or equal 2. Let σ be a permutation of $\mathbb{N}$. Assume that, for some n, m in $\mathbb{N}$, $\sigma[C_n^i(k)] \in \mathcal{Z}(m,j)$ for every i in $\{1,\ldots,k^n\}$. Let $Z \in \mathcal{Z}(n,k)$. Then $\sigma[Z] \in \mathcal{Z}(m,j)$.

Proof: The proof of R32.5.1 works with any fixed $j \geq 2$ for the images, not just $j = k$. To make that observation clear, the adjusted proof follows. Finite subsets of $\mathbb{N}$ are in $\mathcal{Z}(m,j)$, being associated with $\{1,\ldots,j^m\}$. By definition $Z \in \mathcal{Z}(n,k)$ implies that, for every $i \in \{1,\ldots,k^n\}$, either $A_i = Z \cap C_n^i(k)$ is finite or $B_i = (\mathbb{N} - Z) \cap C_n^i(k)$ is finite. Note that $C_n^i(k) = A_i \cup B_i$ and $A_i \cap B_i = \emptyset$ so that $A_i = C_n^i(k) \cap (\mathbb{N} - B_i)$. It will be shown that $\sigma[A_i]$ is in $\mathcal{Z}(m,j)$ for all i. When A_i is finite, $\sigma[A_i]$ is in $\mathcal{Z}(m,j)$. When A_i is not finite, $\sigma[B_i] \in \mathcal{Z}(m,j)$, as is its complement. By hypothesis $\sigma[C_n^i(k)]$ is in $\mathcal{Z}(m,j)$. Now $\sigma[A_i] = \sigma[C_n^i(k)] \cap (\mathbb{N} - \sigma[B_i])$ because σ is a permutation. Since $\mathcal{Z}(m,j)$ is closed under finite intersections, $\sigma[A_i]$, is also in $Z(m,j)$. Since $Z = \cup_{i=1}^{k^n} A_i$ and a normal basis is closed under finite unions, $\sigma[Z] = \cup_{i=1}^{k^n} \sigma[A_i]$ is in $\mathcal{Z}(m,j)$.

As in [7] $\mathbb{N}_\infty$ denotes the Wallman compactification generated by $\mathcal{Z}_\infty$, a normal basis for $\mathbb{N}$ with the discrete topology. $\mathcal{Z}_\infty$ is $\cup_{k=2}^\infty \mathcal{Z}_k$, which is closed under complementation. The following is similar to R32.5.3.

Proposition R32.Add.12 Let σ be a permutation of $\mathbb{N}$. Then σ extends to an auto-homeomorphism of $\mathbb{N}_\infty$ if and only if $\sigma[C_n^j(k)]$ and $\sigma^{-1}[C_n^j(k)]$ are both in $\mathcal{Z}_\infty$ for every $n, j, k \in \mathbb{N}$ with $k \geq 2$.

Proof: First assume σ extends. By R32.4.7 $\sigma[Z]$ and $\sigma^{-1}[Z]$ are in $\mathcal{Z}_\infty$ for every $Z \in \mathcal{Z}_\infty$. The condition follows because for every $n, j, k \in \mathbb{N}$ with $k \geq 2$, $C_n^j(k) \in \mathcal{Z}_\infty$. Conversely assume $\sigma[C_n^j(k)]$ and $\sigma^{-1}[C_n^j(k)]$ are both in $\mathcal{Z}_\infty$ for every $n, j, k \in \mathbb{N}$ with $k \geq 2$ and let $Z \in \mathcal{Z}_\infty$. There is $l \in \mathbb{N}$ with $l \geq 2$ such that $Z \in \mathcal{Z}_l$. By definition $Z \in \mathcal{Z}(p,l)$ for some $p \in \mathbb{N}$. For every j in $\{1,\ldots,l^p\}$, $\sigma[C_p^j(l)] \in \mathcal{Z}_\infty$ by assumption. Because $\{\mathcal{Z}_k : k \geq 2\}$ is a directed set under containment, there is $r \in \mathbb{N}$ such that for every j in $\{1,\ldots,l^p\}$, $\sigma[C_p^j(l)] \in \mathcal{Z}_r$. Because $\mathcal{Z}_r = \cup_{i=2}^\infty \mathcal{Z}(i,r)$ and $\mathcal{Z}(i,r) \subseteq \mathcal{Z}(i+1,r)$ for all i, there is $q \geq 2$ such that for every j in $\{1,\ldots,l^p\}$, $\sigma[C_p^j(l)] \in \mathcal{Z}(q,r)$. By R32.Add.11 $\sigma[Z] \in \mathcal{Z}(q,r) \subseteq \mathcal{Z}_r \subseteq \mathcal{Z}_\infty$. The same argument for the permutation σ^{-1} shows that $\sigma^{-1}[Z] \in \mathcal{Z}_\infty$ for every $Z \in \mathcal{Z}_\infty$. It now follows from R32.4.3 that σ extends to an auto-homeomorphism of $\mathbb{N}_\infty$.

Recall from [10] that $\mathcal{D}_\infty = \cup_{k=2}^\infty \mathcal{D}_k$ is a normal basis for $(\mathbf{Z}\ , \tau_\infty)$, where $\tau_\infty = \vee_{k=2}^\infty \tau_k$. R27.3.5 shows that $(\omega(\mathcal{D}_\infty), \iota_{\mathcal{D}_\infty})$ is equivalent to $(\mathbf{R}_\infty, f_\infty)$. $\mathcal{D}_\infty$ is closed under complementation because each $\mathcal{D}_k$ is.

Lemma R32.Add.13 Let h be a permutation of $\mathbf{Z}$. Then h extends (relative to the embedding $\iota_{\mathcal{D}_\infty}$) to a homeomorphism of $\omega(\mathcal{D}_\infty)$ if and only if $h[D_n^z(k)]$ and $h^{-1}[D_n^z(k)]$ are both in $\mathcal{D}_\infty$ for every $z \in \mathbf{Z}$, $n, k \in \mathbb{N}$ with $k \geq 2$.

Proof: If h extends, the condition follows from R34.4.7 since $D_n^z(k)$ is in $\mathcal{D}_\infty$ for every $z \in \mathbf{Z}$, $n, k \in \mathbb{N}$ with $k \geq 2$. For the converse assume the condition holds and let $D \in \mathcal{D}_\infty$.

By definition of $\mathcal{D}_\infty$ there is $k \in \mathbb{N}$ with $k \geq 2$ such that $D \in \mathcal{D}_k$. By definition of $\mathcal{D}_k$, D is a finite union of equivalence classes, i.e., there exist $t \in \mathbb{N}$, $z(1) \dots, z(t) \in \mathbf{Z}$, and $n(1), \dots, n(t) \in \mathbb{N}$ such that $D = \cup_{i=1}^t D_{n(i)}^{z(i)}(k)$. Since $\mathcal{D}_k$ is closed under finite unions, $h[D] = \cup_{i=1}^t h[D_{n(i)}^{z(i)}(k)]$ and $h^{-1}[D] = \cup_{i=1}^t h^{-1}[D_{n(i)}^{z(i)}(k)]$ are both in $\mathcal{D}_k \subseteq \mathcal{D}_\infty$. It follows from R32.4.3 that h extends as required.

The following is similar to R32.5.6.

Proposition R32.Add.14 Let h be a permutation of $\mathbf{Z}$. Then h extends (relative to the embedding f_∞) to a homeomorphism of $\mathbf{R}_\infty$ if and only if $h[D_n^z(k)]$ and $h^{-1}[D_n^z(k)]$ are both in $\mathcal{D}_\infty$ for every $z \in \mathbf{Z}$, $n, k \in \mathbb{N}$ with $k \geq 2$.

Proof: This follows from the previous lemma, R32.1.3, and R27.3.5.

Recall from [10] that $\mathcal{C}_\infty = \cup_{k=2}^\infty \mathcal{C}_k$ is a normal basis for ($\mathbb{N}$, σ_∞), where σ_∞ is the relative topology on $\mathbb{N}$ from τ_∞. R27.4.15 shows that $(\omega(\mathcal{C}_\infty), \iota_{\mathcal{C}_\infty})$ is equivalent to ($\mathbf{R}_\infty$, g_∞), where g_∞ is the restriction of f_∞ to $\mathbb{N}$. $\mathcal{C}_\infty$ is closed under complementation because each $\mathcal{C}_k$ is.

Lemma R32.Add.15 Let h be a permutation of $\mathbf{Z}$. Then h extends (relative to the embedding $\iota_{\mathcal{C}_\infty}$) to a homeomorphism of $\omega(\mathcal{C}_\infty)$ if and only if $h[C_n^j(k)]$ and $h^{-1}[C_n^j(k)]$ are both in $\mathcal{C}_\infty$ for every $n, j, k \in \mathbb{N}$ with $k \geq 2$.

Proof: If h extends, the condition follows from R34.4.7 since $C_n^j(k)$ is in $\mathcal{C}_\infty$ for every $n, j, k \in \mathbb{N}$ with $K \geq 2$. Conversely assume the condition holds and let $C \in \mathcal{C}_\infty$. Then $C = D \cap \mathbb{N}$ for some $D \in \mathcal{D}_\infty$. By definition of $\mathcal{D}_\infty$ there is $k \in \mathbb{N}$ with $k \geq 2$ such that $D \in \mathcal{D}_k$. By definition of $\mathcal{D}_k$, D is a finite union of equivalence classes, i.e., there exist t in $\mathbb{N}$, $z(1) \dots, z(t) \in \mathbf{Z}$, and $n(1), \dots, n(t) \in \mathbb{N}$ such that $D = \cup_{i=1}^t D_{n(i)}^{z(i)}(k)$. For each i pick $j(i) \in \mathbb{N}$ such that $j(i) \equiv z(i) \bmod k^{n(i)}$. Since $D_{n(i)}^{z(i)}(k) \cap \mathbb{N} = C_{n(i)}^{j(i)}(k)$, $C = \cup_{i=1}^t C_{n(i)}^{j(i)}(k)$. Since $\mathcal{C}_\infty$ is closed under finite unions, by the hypothesis for this part $h[C] = \cup_{i=1}^t h[C_{n(i)}^{j(i)}(k)]$ and $h^{-1}[C] = \cup_{i=1}^t h^{-1}[C_{n(i)}^{j(i)}(k)]$ are both in $\mathcal{C}_\infty$. By R32.4.3 h extends as required.

The following is similar to R32.5.8.

Proposition R32.Add.16 Let h be a permutation of $\mathbb{N}$. Then h extends (relative to the embedding g_∞) to a homeomorphism of $\mathbf{R}_\infty$ if and only if $h[C_n^j(k)]$ and $h^{-1}[C_n^j(k)]$ are both in $\mathcal{C}_\infty$ for every $n, j, k \in \mathbb{N}$ with $k \geq 2$.

Proof: This follows from the previous lemma, R32.1.3, and R27.4.14.

The following is similar to R32.5.9.

Corollary R32.Add.17 Let h be a permutation of $\mathbb{N}$ and assume h extends (relative to g_∞) to a homeomorphism of $\mathbf{R}_\infty$. Then h extends to a homeomorphism of $\mathbb{N}_\infty$.

Proof: Because $\mathcal{C}_\infty \subseteq \mathcal{Z}_\infty$, this follows from R32.Add.16 and R32.Add.12.

The example in R32.5.10 also shows that the converse of R32.Add.17 is false.

Example R32.Add.18 Let h be the permutation of $\mathbb{N}$ described in R32.5.10. As shown there, for $n, j, k \in \mathbb{N}$ with $k \geq 2$, $h[C_n^j(k)] \in \mathcal{Z}_k \subseteq \mathcal{Z}_\infty$. Because $h^{-1} = h$, by R32.Add.12 h extends to an auto-homeomorphism of $\mathbb{N}_\infty$. It was also shown in R35.5.10 that 2 is the only even number in $h[C_1^1(2)]$. Since any equivalence class containing 2 must contain infinitely many evens, $h[C_1^1(2)]$ is not a finite union of equivalence classes, i.e., it is not in $\mathcal{C}_\infty$. By R32.Add.16, h does not extend (relative to g_∞) to $\mathbf{R}_\infty$.

Note that it can be shown that h in the previous example is not a homeomorphism of $(\mathbb{N}, \sigma_\infty)$ or of $(\mathbb{N}, \sigma_k)$ for any $k \geq 2$.

Proof of the last claim: For every $k \geq 2$, $\mathcal{B}_k^* = \{B \cap \mathbb{N} : B \in \mathcal{B}_k\}$ is a basis for σ_k. First suppose k is even. Then $C_1^1(k)$ contains only odd integers and 2 is the only even integer in $h[C_1^1(k)]$. For every n, $C_n^2(k)$ contains infinitely many evens. Thus no basic set containing 2 is a subset of $h[C_1^1(k)]$, i.e., $h[C_1^1(k)]$ is not in σ_k, i.e., h is neither open nor continuous from $(\mathbb{N}, \sigma_k)$ to itself. Now suppose k is odd. Now the even members of $h[C_1^1(k)]$ are 2 and $\{1 + mk : m$ is an odd positive integer $\}$. Suppose $C_n^2(k) \subseteq h[C_1^1(k)]$. Then $2+2k^n = 1+mk$ for some odd m. That implies that k is a divisor of 1, a contradiction since $k \geq 2$. As before, no basic set containing 2 is a subset of $h[C_1^1(k)]$, i.e., $h[C_1^1(k)]$ is not in σ_k, i.e., h is neither open nor continuous from $(\mathbb{N}, \sigma_k)$ to itself. Finally, $\cup_{k=2}^\infty \mathcal{B}_k^*$ is a subbase for σ_∞. Any finite intersection of equivalence classes containing 2 must contiain infinitely many even numbers. Thus $h[C_1^1(2)]$ is not in σ_∞ because no basic set containing 2 is a subset of $h[C_1^1(2)]$, i.e., h is not open or continuous from $(\mathbb{N}, \sigma_\infty)$ to itself.

An unanswered question: With the additional assumption that h is a homeomorphism (of $(\mathbb{N}, \sigma_k)$, respectively $(\mathbb{N}, \sigma_\infty)$), can partial converses of R32.5.9, respectively R32.Add.17, be proven?

Added 2025

This added subsection records some generalizations, which do not seem to lead to further development.

Theorem R32.Add.19 Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be separated, totally bounded uniform spaces. Let (C, f) and (D, g) be T_2-compactifications in the compactification classes corresponding to $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $h : X \to Y$ be onto. The following are equivalent.

i) h extends to a continuous one-to-one map from C onto D.

ii) h extends to a homeomorphism from C onto D.

iii) $h : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a unimorphism.

Proof: Assume i). Because C is compact and D is T_2, the given bijective continuous extension must be a closed map and so a homeomorphism. Now assume ii) and let $H : C \to D$ be the homeomorphism which extends h. By the definition of an extension $H \circ f = g \circ h$. Thus h is one-to-one and H^{-1} is a continuous extension of h^{-1}. By R7.Add.7 both $h : (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $h^{-1} : (Y, \mathcal{V}) \to (X, \mathcal{U})$ are uniformly continuous, i.e., h is a unimorphism. Finally, assume iii). By R7.1.3 there are continuous extensions $H : C \to D$ of h and $G : D \to C$ of h^{-1}. By the definition of extension, the continuous maps $G \circ H$ and Id_C agree on the dense subset $f[X]$. Since Y is T_2, $G \circ H = \mathrm{Id}_C$ and so H is one-to-one. Similarly, $H \circ G = \mathrm{Id}_D$ so that H is onto. Thus i) holds.

Note that R32.1.2 is a special case of the above. Although the intent of this section was to consider the case of auto-homeomorphisms, the result as stated above does not assume that h is a homeomorphism of the underlying topological spaces.

Next the previous result is rewritten with a more topological emphasis, although some uniform concepts are unavoidable.

Corollary R32.Add.20 Let (X, τ) and (Y, σ) be $T_{3\frac{1}{2}}$ spaces. Let (C, f) and (D, g) be T_2-compactifications of (X, τ) and (Y, σ) respectively. Let $\mathcal{U}$ and $\mathcal{V}$ be separated, totally bounded uniformities for X and Y respectively such that (C, f) is in the compactification class corresponding to $\mathcal{U}$ and (D, g) is in the compactification class corresponding to $\mathcal{V}$.

Let $h : X \to Y$ be onto. Then the following are equivalent.

i) h extends to a continuous one-to-one map from C onto D.

ii) h extends to a homeomorphism from C onto D.

iii) $h : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a unimorphism.

Proof: This follows easily from R32.Add.19.

Extension of a one-to-one map which is not onto basically involves changing the target space to the image. The routine details are presented in the following 8 items. The first definition introduces a non-standard notation for a perhaps pedantic distinction.

Definition R32.Add.21 Let A, B be sets and $m : A \to B$ a function.
Let $m[A] \subseteq T \subseteq B$. Define ${}^T m : A \to T$ by ${}^T m(a) = m(a)$.

For a topological space (Y, σ) and $S \subseteq Y$, σ_S denotes the subspace topology on S from σ.

Lemma R32.Add.22 Let (X, τ) and (Y, σ) be topological spaces and let $m : X \to Y$.
Let $m[X] \subseteq T \subseteq Y$.
Then $m : (X, \tau) \to (Y, \sigma)$ is continuous if and only if ${}^T m : (X, \tau) \to (T, \sigma_T)$ is continuous.

Proof: It is easy to check that, for $G \subseteq Y$, $({}^T m)^{-1}[G \cap T] = m^{-1}[G]$. The conclusion follows.

For a uniform space $(Y, \mathcal{V})$ and $S \subseteq Y$, $\mathcal{V}_S$ denotes the subspace uniformity on S induced by $\mathcal{V}$. It is known that $\tau(\mathcal{V}_S)$ is the subspace topology on S from $\tau(\mathcal{V})$.

Lemma R32.Add.23 Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces and let $m : X \to Y$. Let $m[X] \subseteq T \subseteq Y$. Then $m : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous if and only if ${}^T m : (X, \mathcal{U}) \to (T, \mathcal{V}_T)$ is uniformly continuous.

Proof: It is easy to check that, for $V \subseteq Y \times Y$, $({}^T m \times {}^T m)^{-1}[V \cap (T \times T)] = (m \times m)^{-1}[V]$. The conclusion follows.

For a topological space D, c_D denotes the closure in D.

Lemma R32.Add.24 Let (X, τ) and (Y, σ) be $T_{3\frac{1}{2}}$ spaces. Let (C, f) and (D, g) be T_2-compactifications of (X, τ) and (Y, σ) respectively. Let $\phi : X \to Y$ have a continuous extension $\Phi : C \to D$. Then $\Phi[C] = c_D(g[\phi[X]])$.

Proof: By the definition of an extension, $\Phi \circ f = g \circ \phi$ so that $g[\phi[X]] \subseteq \Phi[C]$. Since Φ is continuous, C is compact, and D is T_2, $\Phi[C]$ is closed in D and so $c_D(g[\phi[X]]) \subseteq \Phi[C]$. Now let $c \in C$. By the density of $f[X]$, there is a net $\{x_\alpha\}$ in X such that $f(x_\alpha) \to c$. By continuity $\Phi(f(x_\alpha)) \to \Phi(c)$. Since $\Phi(f(x_\alpha)) = g(\phi(x_\alpha))$, $g(\phi(x_\alpha)) \to \Phi(c)$. Thus $\Phi(c) \in c_D(g[\phi[X]])$ and so $\Phi[C] \subseteq c_D(g[\phi[X]])$.

For the rest of this added subsection, the notation of R32.Add.21 will be avoided in the statement of results by referring the treatment of a function as a map into some suitable subset of the image space. For example, in the next two lemmas, the phrase "$g|_S$ is treated as a mapping into $c_D(g[S])$" means ${}^T g|_S$ with $T = c_D(g[S])$ is being considered.

Lemma R32.Add.25 Let (Y, σ) be a $T_{3\frac{1}{2}}$ space and (D, g) a T_2-compactification of Y. Let $S \subseteq Y$. Then $(c_D(g[S]), g|_S))$ is a T_2-compactification of (S, σ_S), where $g|_S$ is treated as a mapping into $c_D(g[S])$.

Proof: Because D is compact and T_2, so is the closed subspace $c_D(g[S])$. Now write $T = c_D(g[S])$. Clearly ${}^T g|_S[S] = g[S]$, which is dense in its closure. As the restriction of a continuous one-to-one map, ${}^T g|_S$ is continuous and one-to-one. An element of σ_S has the form $G \cap S$ where $G \in \sigma$. By hypothesis there is O open in D such that $g[G] = O \cap g[X]$.

Then ${}^T g|_S[G \cap S] = O \cap g[S]$, which is open in the relative topology on $g[S]$. The conclusion follows.

Note that R27.4.1 is a special case of the last lemma.

Lemma R32.Add.26 Let (Y, σ) be a $T_{3\frac{1}{2}}$ space and (D, g) a T_2-compactification of Y. Let $\mathcal{V}$ be the separated totally bounded uniformity for Y corresponding to the compactification class of (D, g). Let $S \subseteq Y$. Then $\mathcal{V}_S$ is the separated totally bounded uniformity for Y corresponding to the compactification class of $(c_D(g[S]), g|_S))$, where $g|_S$ is treated as a mapping into $c_D(g[S])$.

Proof: Let $\mathcal{W}$ be the unique uniformity for the compact D so that $\mathcal{W}_{c_D(g[S])}$ is the unique uniformity for $c_D(g[S])$. Again write $T = c_D(g[S])$. By R1.6a $\mathcal{V}$ is the uniformity making g a uniform embedding into D and it is sufficient to show that $\mathcal{V}_S$ is the uniformity making ${}^T g|S$ a uniform embedding into $c_D(g[S])$. Since g is one-to-one and uniformly continuous, ${}^T g|S$ also has those properties by previous lemmas. A typical element of $\mathcal{V}_S$ has the form $(S \times S) \cap V$, where $V \in \mathcal{V}$. Since g is a uniform embedding, there is $W \in \mathcal{W}$ such $(g \times g)[V] = (g[X] \times g[X]) \cap W$. Because g is one-to-one, it is easy to verify that

$$({}^T g|_S \times {}^T g|_S)[(S \times S) \cap V] = (g[S] \times g[S]) \cap (W \cap (c_D(g[S]) \times c_D(g[S])).$$

Thus ${}^T g|_S$ is a uniform embedding into $c_D(g[S])$ and the conclusion follows.

Lemma R32.Add.27 Let (X, τ) and (Y, σ) be $T_{3\frac{1}{2}}$ spaces. Let (C, f) and (D, g) be T_2-compactifications of (X, τ) and (Y, σ) respectively. Let $\phi : X \to Y$ have a continuous extension $\Phi : C \to D$. If Φ and $g|_{\phi[X]}$ are treated as a mappings into $c_D(g[\phi[X]])$, Φ is the continuous extension of ϕ (treated as a map into $\phi[X]$) from C to $c_D(g[\phi[X]])$.

Proof: By the definition of an extension, $\Phi \circ f = g \circ \phi$. Because of R32.Add.24, with $T = c_D(g[\phi[X]])$, ${}^T\Phi$ is defined and by R32.Add.22 it is continuous. The given equation yields ${}^T\Phi \circ f = {}^T g|_{\phi[X]} \circ {}^{\phi[X]}\phi$, i.e., the conclusion holds.

Proposition R32.Add.28 Let (X, τ) and (Y, σ) be $T_{3\frac{1}{2}}$ spaces. Let (C, f) and (D, g) be T_2-compactifications of (X, τ) and (Y, σ) respectively. Let $\mathcal{U}$ and $\mathcal{V}$ be separated, totally bounded uniformities for X and Y respectively such that (C, f) is in the compactification class corresponding to $\mathcal{U}$ and (D, g) is in the compactification class corresponding to $\mathcal{V}$. Let $\phi : X \to Y$ be one-to-one. Then the following are equivalent:

i) ϕ has a continuous, one-to-one extension from C to D.

ii) $\phi : (X, \mathcal{U}) \to (\phi[X], \mathcal{V}_{\phi[X]})$ is a unimorphism, where ϕ is treated as a mapping into $\phi[X]$.

iii) $\phi : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a uniform embedding.

Proof: Parts ii) and iii) are equivalent by the definition of uniform embedding. Assume i) holds and let Φ be the continuous one-to-one extension. Let $T = c_D(g[\phi[X]])$, the image of Φ. ${}^T\Phi$ is onto, clearly one-to-one, and, by R32.Add.27, a continuous extension of ${}^{\phi[X]}\phi$. It follows from R32.Add.26 and R32.Add.20 that ${}^{\phi[X]}\phi : (X, \mathcal{U}) \to (\phi[X], \mathcal{V}_{\phi[X]})$ is a unimorphism, i.e., ii) holds. Conversely, assume ii). By R32.Add.26 and R32.Add.19 there is a homeomorphism $\Psi : C \to c_D(g[\phi[X]])$ which extends ${}^{\phi[X]}\phi$, i.e, $\Psi \circ f = g|_{\phi[X]} \circ {}^{\phi[X]}\phi$, where $g|_{\phi[X]}$ is treated as a map into $c_D(g[\phi[X]])$. Define $\Phi : C \to D$ by $\Phi(x) = \Psi(x)$ for all $x \in X$. For $T = c_D(g[\phi[X]])$, ${}^T\Phi = \Psi$ and, since Ψ is one-to-one, Φ is also. By R32.Add.22 Φ is continuous. The given extension equation implies $\Phi(f(x)) = \Psi(f(x)) = g(\phi(x))$ and so Φ extends ϕ. Thus i) holds.