

## The Remnant Rings Are Homeomorphic

In [6] the remnant rings  $\mathbf{R}_k$  (where  $k \geq 2$  is in  $\mathbf{N}$ ) were shown to be compactifications of the integers  $\mathbf{Z}$  with certain non-discrete topologies, which vary with  $k$ . It will first be shown here that, for relatively prime  $a, b$ ,  $\mathbf{R}_a$  and  $\mathbf{R}_b$  are not equivalent compactifications in the generalized sense of [5].

The main result is that the topological space  $\mathbf{R}_2$  is homeomorphic to  $\mathbf{R}_k$  for all  $k \geq 2$  in  $\mathbf{N}$ . Since, for  $p$  prime,  $\mathbf{R}_p$  is topologically isomorphic to the  $p$ -adic integers by R17.1.19, this is a slight generalization of the known result that, for  $p, q$  prime, the  $p$ -adic integers are homeomorphic to the  $q$ -adic integers. The proof of the main result is based on the presentation of that fact in exercises 12 and 13, p. 65, of [1].

The remnant rings are also shown to be homeomorphic to the Cantor set and some results are derived on the question of when  $\mathbf{N}_j$  and  $\mathbf{N}_k$  are homeomorphic topological spaces.

The first proposition makes use of the following notation from [6]: Let  $k \geq 2$  be in  $\mathbf{N}$ . For  $n \in \mathbf{N}$  and  $x \in \mathbf{Z}$ ,  $D_n^x(k)$  is the equivalence class in  $\mathbf{Z}$  of  $x \bmod k^n$ . In [6] it is shown that  $\{D_n^x(k) : n \in \mathbf{N} \text{ and } x \in \mathbf{Z}\}$  is a clopen basis for a topology  $\tau_k$  on  $\mathbf{Z}$  and that  $\mathbf{R}_k$  with a suitable embedding is a compactification of  $(\mathbf{Z}, \tau_k)$ .

**Proposition R26.1** Let  $a, b \geq 2$  be in  $\mathbf{N}$  with  $(a, b) = 1$ . Then  $\mathbf{R}_a \not\cong \mathbf{R}_b$ .

Proof: By R13.1.5i it is sufficient to show that  $\tau_a \not\subseteq \tau_b$ , which will be done by showing  $D_1^1(a)$  is not in  $\tau_b$ . Deny. Then there are  $n \in \mathbf{N}$  and  $x \in \mathbf{Z}$  such that  $1 \in D_n^x(b) \subseteq D_1^1(a)$ . Since these are equivalence classes,  $D_n^x(b) = D_n^1(b) \subseteq D_1^1(a)$ . Then  $1 + b^n \equiv 1 \pmod{a}$  so that  $b^n \equiv 0 \pmod{a}$ , i.e.,  $a$  divides  $b^n$ , which contradicts the hypothesis that  $a$  and  $b$  are relatively prime.

Comment: In the last proposition the roles of  $a$  and  $b$  could be reversed thereby showing that the compactifications  $\mathbf{R}_a$  and  $\mathbf{R}_b$  are not related in either direction and, of course, are not equivalent. In R10.3.5 it was shown that, for two distinct primes  $p$  and  $q$ ,  $\mathbf{N}_p \not\cong \mathbf{N}_q$ , a fact which could easily be generalized to a relatively prime pair. The proof of R10.3.5 could be modified to verify R26.1, but the approach used in R26.1 can't be used to prove R10.3.5, since  $\mathbf{N}_p$  and  $\mathbf{N}_q$  are both compactifications of  $\mathbf{N}$  with the discrete topology.

Before presenting the main result, some needed properties of the derived sequence of an element in  $\mathbf{R}_k$  for  $k \geq 2$  will be developed. Recall that each element of  $\mathbf{R}_k$  determines two related but distinct sequences: the derived sequence (defined in R20.1) of the form  $\{a_n\}_{n=0}^\infty$ , where  $a_n \in \{0, \dots, k-1\}$  for all  $n$ , and the associated sequence (defined in R10.2.3) of the form  $\{x_n\}_{n=1}^\infty$ , where  $x_n \in \{1, \dots, k^n\}$  for all  $n$ .

**Lemma R26.2** Let  $\{a_n\}_{n=0}^\infty$  be a sequence with each  $a_n$  in  $\{0, \dots, k-1\}$ , where  $k \geq 2$  is in  $\mathbf{N}$ . Then there is  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  in  $\mathbf{R}_k$  such that the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is  $\{a_n\}_{n=0}^\infty$ .

Proof: First assume  $a_n = 0$  for every  $n$ . Let  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  be the additive identity of  $\mathbf{R}_k$ , which has associated sequence  $\{k^n\}_{n=1}^\infty$ . By definition the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is  $\{a_n\}_{n=0}^\infty$ .

Now assume there is  $n$  with  $a_n \neq 0$  and let  $m$  be the smallest of  $\{n : a_n \neq 0\}$ . If  $m = 0$ , let  $x_1 = a_0$  and let  $x_{n+1} = x_n + a_n k^n$  for  $n \geq 1$ . By R10.2.6 there is a unique  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  in  $\mathbf{R}_k$  which is associated with  $\{x_n\}_{n=1}^\infty$ . Let  $\{b_n\}_{n=0}^\infty$  be the derived

sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$ . Since  $x_1 = a_0 \neq k$ , by definition  $b_0 = x_1 = a_0$  and, for  $n \geq 1$ ,  $b_n = (x_{n+1} - x_n)/k^n = a_n$ . Thus the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is  $\{a_n\}_{n=0}^\infty$ .

As a final case assume  $m \geq 1$ . Let  $x_i = k^i$  for  $1 \leq i \leq m$ , let  $x_{m+1} = k^m a_m$ , and let  $x_{i+1} = x_i + a_i k^i$  for  $i \geq m+1$ . Note that, for  $1 \leq i \leq m-1$ ,  $x_{i+1} = k^i + (k-1)k^i = x_i + (k-1)k^i$ . Also, since  $a_m \in \{1, \dots, k-1\}$ ,  $x_{m+1} = k^m + (a_m - 1)k^m = x_m + (a_m - 1)k^m$ . Thus R10.2.6 applies: there is  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  in  $\mathbf{R}_k$  such that  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is associated with  $\{x_n\}_{n=1}^\infty$ . Let  $\{b_n\}_{n=0}^\infty$  be the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$ . Note that the smallest of  $\{n : x_n \neq k^n\}$  is  $m+1$  and so  $b_i = 0 = a_i$  for  $0 \leq i \leq m-1$ . Also by definition,  $b_m = x_{m+1}/k^m = a_m$ . Finally, for  $i \geq m+1$ , by definition  $b_i = (x_{i+1} - x_i)/k^i = a_i$ . Thus  $b_n = a_n$  for all  $n$ , i.e., the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is  $\{a_n\}_{n=0}^\infty$ .

**Lemma R26.3** Let  $k \geq 2$  in  $\mathbf{N}$ , let  $\mathcal{F}$  be in  $\mathbf{R}_k$ , and let  $\{a_n\}_{n=0}^\infty$  be the derived sequence of  $\mathcal{F}$ . Then  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}$ .

Proof: Let  $\{y_n\}_{n=1}^\infty$  be the associated sequence of  $\mathcal{F}$ . If  $y_n = k^n$  for every  $n$ , then  $\mathcal{F}$  is the additive identity of  $\mathbf{R}_k$  as shown in [4], especially R12.5.9. By definition  $a_n = 0$  for every  $n$  and by the construction in the previous lemma,  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}$ . Now assume that  $y_n \neq k^n$  for some  $n$  and let  $l$  be the smallest of  $\{n : y_n \neq k^n\}$ . Let  $\{x_n\}_{n=1}^\infty$  be the sequence associated with  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$ , as constructed in the proof of R26.2. If  $l = 1$ , by R20.1  $a_0 = y_1$  and  $a_n = (y_{n+1} - y_n)/k^n$  for  $n \geq 1$ . Since  $y_1 \neq 0$ , by construction,  $x_1 = a_0 = y_1$  and  $x_{n+1} = x_n + a_n k^n$  for all  $n \geq 1$ . Induction and routine algebra show that  $x_{n+1} = y_{n+1}$  for  $n \geq 1$ . By R10.2.4  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}$ . Finally, suppose  $l > 1$ . By R20.1  $a_0 = a_1 = \dots = a_{l-2} = 0$ ,  $a_{l-1} = y_l/k^{l-1}$ , and, for  $n \geq l$ ,  $a_n = (y_{n+1} - y_n)/k^n$ . The smallest of  $\{n : a_n \neq 0\}$  is  $l-1$ . By construction,  $x_n = k^n$  for  $1 \leq n \leq l-1$ ,  $x_l = k^{l-1} a_{l-1}$ , and, for  $n \geq l$ ,  $x_{n+1} = x_n + a_n k^n$ . By the choice of  $l$ ,  $y_n = k^n = x_n$  for  $1 \leq n \leq l-1$ . Also  $x_l = k^{l-1}(y_l/k^{l-1}) = y_l$ . For  $n \geq l$ , induction and routine algebra show  $x_{n+1} = y_{n+1}$ . Again by R10.2.4  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}$ .

**Corollary R26.4** Let  $k \geq 2$  in  $\mathbf{N}$  and let  $\mathcal{F}$  and  $\mathcal{G}$  be in  $\mathbf{R}_k$ . Let  $\{a_n\}_{n=0}^\infty$  and  $\{b_n\}_{n=0}^\infty$  be the derived sequences of  $\mathcal{F}$  and  $\mathcal{G}$  respectively. Then  $\mathcal{F} = \mathcal{G}$  if and only if  $a_n = b_n$  for all  $n \geq 0$ .

Proof: If  $\mathcal{F} = \mathcal{G}$ , each has the same associated sequence and so the same derived sequence. If  $a_n = b_n$  for every  $n$ , then  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}(\{b_n\}_{n=0}^\infty)$  and so By R26.3  $\mathcal{F} = \mathcal{G}$ .

Next some terminology and notation used in Robert's hint will be presented: A word is a finite string with characters from  $\{0, 1\}$  or, formally, a function from an initial segment of  $\mathbf{N}$  into  $\{0, 1\}$ . Note that the function from  $\emptyset$  is allowed. It will be called the null word or  $\eta$ . Each word has a length, which is its number of characters, i.e., the number of natural numbers in its domain. The length of  $\eta$  is 0. With the operation of concatenation,  $M_2$ , the set of all words, is a non-commutative semi-group with unit  $\eta$ , i.e., a monoid. An infinite word is a sequence in  $\{0, 1\}$ . If  $w$  denotes the infinite word  $\{t_i\}_{i=1}^\infty$ ,  $w_0 = \eta$  and  $w_n$  is the word  $\{t_i\}_{i=1}^n$ . For an infinite word  $w$ ,  $w \notin M_2$  but  $w_n \in M_2$  for all  $n$  and the length of  $w_n$  is  $n$ . Usually string notation will be used instead of the more formal functional representation, e.g., 101 will denote the word  $\{t_i\}_{i=1}^3$  with  $t_1 = 1$ ,  $t_2 = 0$ , and  $t_3 = 1$ .

**Definition R26.5** Define  $d$  from  $\mathbf{R}_2$  to the set of infinite words by  $d(\mathcal{F}) = \{a_n\}_{n=0}^\infty$ , where  $\{a_n\}_{n=0}^\infty$  is the derived sequence of  $\mathcal{F}$ .

Note: If one uses string notation, because the derived sequence indexing begins at 0,  $d(\mathcal{F})_0 = \eta$ ,  $d(\mathcal{F})_1 = a_0$ , and  $d(\mathcal{F})_{n+1} = d(\mathcal{F})_n a_n$ .

**Lemma R26.6** The map  $d$  is a bijection.

Proof: Each  $\mathcal{F}$  in  $\mathbf{R}_2$  has a unique associated sequence and so a unique derived sequence. Thus  $d$  is indeed a function. R26.2 shows that  $d$  is onto and R26.4 shows that  $d$  is one-to-one.

**Lemma R26.7** Let  $k \geq 2$  and  $\mathcal{F}, \mathcal{G}$  be in  $\mathbf{R}_k$  with associated sequences  $\{x_n\}_{n=1}^\infty$  and  $\{y_n\}_{n=1}^\infty$  respectively. If  $x_m = y_m$  for some  $m$ , then  $x_j = y_j$  for  $1 \leq j \leq m$ .

Proof: By induction: The conclusion clearly holds for  $m = 1$ . Assume it holds for  $m$  and suppose  $x_{m+1} = y_{m+1}$ . By R10.2.5i  $x_{m+1} = x_m + sk^m$  and  $y_{m+1} = y_m + tk^m$ , where  $s, t \in \{0, 1, \dots, k-1\}$ . Then  $x_m - y_m = (t-s)k^m$ , i.e., if  $s \neq t$ ,  $k^m$  divides  $x_m - y_m$ . Since  $1 \leq x_m, y_m \leq k^m$ ,  $|x_m - y_m| \leq k^m - 1$  and so  $s = t$ , i.e.,  $x_m = y_m$ . By the induction hypothesis, the conclusion holds for  $m+1$ .

**Lemma R26.8** Let  $\mathcal{F}, \mathcal{G}$  be in  $\mathbf{R}_2$  with associated sequences  $\{x_n\}_{n=1}^\infty$  and  $\{y_n\}_{n=1}^\infty$  respectively. If  $x_m = y_m$  for some  $m$ , then  $d(\mathcal{F})_j = d(\mathcal{G})_j$  for  $0 \leq j \leq m$ .

Proof: Let  $\{a_n\}_{n=0}^\infty$  be the derived sequence of  $\mathcal{F}$  and let  $\{b_n\}_{n=0}^\infty$  be the derived sequence of  $\mathcal{G}$ . By the definition of  $d$ , it is sufficient to show  $a_n = b_n$  for  $0 \leq n \leq m-1$ . As a first case, suppose  $x_m = 2^m$ . Since the additive identity for  $\mathbf{R}_2$  has associated sequence  $\{2^n\}_{n=1}^\infty$ , by R26.7  $x_n = 2^n = y_n$  for  $1 \leq n \leq m$ . By definition of the derived sequence  $a_n = b_n = 0$  for  $0 \leq n \leq m-1$ . Now suppose  $x_m \neq 2^m$  and let  $l$  be the smallest of  $\{n : x_n \neq 2^n\}$ . Clearly  $l \leq m$ . Since  $x_m = y_m$ , by R26.7  $l$  is also the smallest of  $\{n : y_n \neq 2^n\}$ . If  $l = 1$ , by definition  $a_0 = x_1 = y_1 = b_0$  and, for  $n \geq 1$ ,  $a_n = (x_{n+1} - x_n)/2^n$  and  $b_n = (y_{n+1} - y_n)/2^n$ . Since  $x_n = y_n$  for  $1 \leq n \leq m$ ,  $a_n = b_n$  for  $0 \leq n \leq m-1$ . If  $l > 1$ , by definition  $a_n = 0 = b_n$  for  $0 \leq n \leq l-2$ ,  $a_{l-1} = x_l/2^{l-1} = y_l/2^{l-1} = b_{l-1}$ , and, for  $n \geq l$ ,  $a_n = (x_{n+1} - x_n)/2^n$  and  $b_n = (y_{n+1} - y_n)/2^n$ . For  $0 \leq n \leq m-1$ , it follows that  $a_n = b_n$ .

**Lemma R26.9** Let  $\{\mathcal{F}_i\}$  be a sequence in  $\mathbf{R}_2$  and let  $\mathcal{F}$  be in  $\mathbf{R}_2$ . Assume  $\mathcal{F}_i \rightarrow \mathcal{F}$ . Then for every  $m \in \mathbf{N}$  there is  $N(m)$  in  $\mathbf{N}$  such that  $i \geq N(m)$  implies  $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$  for  $0 \leq j \leq m$ .

Proof: Fix  $m$  in  $\mathbf{N}$ , let  $\{x_n\}_{n=1}^\infty$  be the associated sequence of  $\mathcal{F}_i$ , and let  $\{x_n\}_{n=1}^\infty$  be the associated sequence of  $\mathcal{F}$ . By R17.2.16 There is  $N(m)$  such that  $i \geq N(m)$  implies  $i x_m = x_m$ . When  $i \geq N(m)$ , R26.8 implies  $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$  for  $0 \leq j \leq m$ .

**Definition R26.10** An auxiliary function for a topological space  $(X, \tau)$  is a map  $\phi$  from  $M_2$  to the power set of  $X$  such that, for every  $w \in M_2$ ,  $\phi(w)$  is closed,  $\phi(\eta) = X$  and, for every  $w \in M_2$ ,  $\phi(w) = \phi(w0) \cup \phi(w1)$ .

**Lemma R26.11** Let  $\phi$  be an auxiliary function for the topological space  $(X, \tau)$  and let  $w$  be an infinite word. Then  $\phi(w_{n+1}) \subseteq \phi(w_n)$  for every  $n \geq 0$ .

Proof; Since  $w_{n+1}$  is either  $w_n0$  or  $w_n1$  and  $\phi(w_n) = \phi(w_n0) \cup \phi(w_n1)$ , the conclusion clearly holds.

Recall that the diameter of a non-empty set in a metric space is the supremum of distances of points in the set. Of course, even for equivalent metrics, the diameter of a given set is not invariant. As an extreme example, in the interval  $(0, 1)$ , the subinterval  $(0, 1/n)$  has diameter  $1/n$  with the usual absolute value metric but, with the equivalent metric  $\rho(x, y) = |1/x - 1/y|$ , it has infinite diameter. The next lemma (undoubtedly known) could be omitted but shows that the subsequent proposition does not depend on a specific metric.

**Lemma R26.12** Let  $\rho$  and  $\sigma$  be equivalent metrics on a set  $X$  and let  $\{A_n\}$  be a sequence of non-empty subsets of  $X$ . Assume  $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ . If  $\text{diam}_{\rho}(A_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\text{diam}_{\sigma}(A_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Proof: Assume  $\text{diam}_{\rho}(A_n) \rightarrow 0$  as  $n \rightarrow \infty$  and let  $a \in \bigcap_{n=1}^{\infty} A_n$ . Let  $\epsilon > 0$ . By definition equivalent metrics generate the same topology and so  $B_{\epsilon}^{\sigma}(a)$  (the open  $\sigma$ -ball of radius  $\epsilon$  centered at  $a$ ) must contain  $B_{\gamma}^{\rho}(a)$  for some  $\gamma > 0$ . By hypothesis there is  $M$  such that, for  $n \geq M$ ,  $\text{diam}_{\rho}(A_n) < \gamma$  and so  $A_n \subseteq B_{\gamma}^{\rho}(a)$ . Thus  $n \geq M$  implies  $\text{diam}_{\sigma}(A_n) \leq 2\epsilon$ .

The next proposition is a version of problem 12 in [1: p.65].

**Proposition R26.13** Let  $\phi$  be an auxiliary function for a compact, metrizable space  $(E, \tau)$ . Assume that, for every word  $\nu$ ,  $\phi(\nu) \neq \emptyset$  and that, for every infinite word  $w$ ,  $\text{diam}(\phi(w_n)) \rightarrow 0$  as  $n \rightarrow \infty$ . Then there is a continuous surjection  $f : \mathbf{R}_2 \rightarrow E$ .

Proof: For any infinite word  $w$ , the collection of closed sets  $\{\phi(w_n) : n \geq 0\}$  has the finite intersection property by hypothesis and R26.11 and so, since  $E$  is compact,  $\bigcap_{n=0}^{\infty} \phi(w_n) \neq \emptyset$ . Suppose  $a \neq b$  are both in  $\bigcap_{n=0}^{\infty} \phi(w_n)$  and let  $\epsilon$  be smaller than the distance from  $a$  to  $b$ . Since  $\text{diam}(\phi(w_n)) \rightarrow 0$  (for any metric generating the topology of  $E$  by R26.12), there is  $n$  such that  $\text{diam}(\phi(w_n)) < \epsilon$ , which contradicts  $a, b \in \phi(w_n)$ . Thus  $\bigcap_{n=0}^{\infty} \phi(w_n)$  is a singleton and the map  $f$  can be defined by letting  $f(\mathcal{F})$  be the unique element of  $\bigcap_{n=0}^{\infty} \phi(d(\mathcal{F})_n)$  for each  $\mathcal{F}$  in  $\mathbf{R}_2$ . To see that  $f$  is onto, let  $e \in E$  and construct a sequence in  $\{0, 1\}$  as follows: Since  $e \in E = \phi(\eta) = \phi(0) \cup \phi(1)$ , pick  $t_1 \in \{0, 1\}$  such  $e \in \phi(t_1)$ . Now assume  $t_1, \dots, t_n$  chosen with  $e \in \phi(\{t_i\}_{i=1}^j)$  for all  $1 \leq j \leq n$ . Since  $e \in \phi(\{t_i\}_{i=1}^n) = \phi(\{t_i\}_{i=1}^n 0) \cup \phi(\{t_i\}_{i=1}^n 1)$ , pick  $t_{n+1}$  in  $\{0, 1\}$  such that  $e \in \phi(\{t_i\}_{i=1}^{n+1})$ . By induction we have an infinite word  $w = \{t_i\}_{i=1}^{\infty}$  such that  $e \in \phi(w_n)$  for every  $n$ . By R26.6 there is  $\mathcal{F}$  in  $\mathbf{R}_2$  such that  $d(\mathcal{F}) = w$ . By definition  $f(\mathcal{F}) = e$ . Lastly, for continuity, let  $\{\mathcal{F}_i\}$  be a sequence in  $\mathbf{R}_2$  and let  $\mathcal{F}$  be in  $\mathbf{R}_2$  with  $\mathcal{F}_i \rightarrow \mathcal{F}$ . It will be shown that  $f(\mathcal{F}_i) \rightarrow f(\mathcal{F})$ . Let  $\epsilon > 0$ . Since  $\text{diam}(\phi(d(\mathcal{F})_n)) \rightarrow 0$  as  $n \rightarrow \infty$  and  $f(\mathcal{F}) \in \phi(d(\mathcal{F})_n)$  for all  $n$ , there is  $m \in \mathbf{N}$  such that  $\phi(d(\mathcal{F})_m)$  is contained in  $B_{\epsilon}(f(\mathcal{F}))$ , the open  $\epsilon$ -ball centered at  $f(\mathcal{F})$ . By R26.9 there is  $N(m)$  in  $\mathbf{N}$  such that  $i \geq N(m)$  implies  $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$  for  $0 \leq j \leq m$ . Then  $i \geq N(m)$  implies  $f(\mathcal{F}_i) \in \phi(d(\mathcal{F}_i)_m) = \phi(d(\mathcal{F})_m) \subseteq B_{\epsilon}(f(\mathcal{F}))$  and the claim is verified. Thus  $f$  is continuous.

The next lemma and R26.16 provide a way to construct auxiliary functions.

**Lemma R26.14** Let  $(X, \tau)$  be a topological space and let  $X = \bigcup_{i=1}^t F_i$ , where each  $F_i$  is a non-empty closed subset and  $t \in \mathbf{N}$ . Then there is a finite set  $D \subseteq M_2$  and  $\psi$  from  $D$  to the non-empty closed subsets of  $X$  such that

- i)  $\eta \in D$  and  $\psi(\eta) = X$ .
- ii) If  $w \in D$  and  $v \in M_2$  with  $\text{length}(v) < \text{length}(w)$ , then  $v \in D$ .
- iii) If  $wv \in D$ , then  $w \in D$  and  $\psi(wv) \subseteq \psi(w)$ .
- iv) There is  $j \geq 1$  such that  $\{w \in M_2 : \text{length}(w) \leq j\} \subseteq D$ .
- v)  $0, 1 \in D$  and  $\psi(0) \cup \psi(1) = X$ .
- vi) If  $w \in D$ , then  $w0 \in D$  if and only if  $w1 \in D$ .
- vii) If  $w$  and  $w0$  are in  $D$ , then  $\psi(w) = \psi(w0) \cup \psi(w1)$ .
- viii) For every  $i$  there is  $w \in D$  such that  $\psi(w) = F_i$  and  $w0 \notin D$ .
- ix) If  $w \in D$  and  $w0 \notin D$ , then  $\psi(w) = F_i$  for some  $i$ .

Proof: If  $t = 1$ , let  $D = \{\eta, 0, 1\}$  and  $\psi(\eta) = \psi(0) = \psi(1) = F_1 = X$ . Now assume

$t \geq 2$  and let  $S_s$  denote  $\{w \in M_2 : \text{length}(w) = s\}$ , where  $s \in \mathbf{N} \cup \{0\}$ . Let  $j$  be the unique non-negative integer such that  $2^j \leq t < 2^{j+1}$ . By induction  $|S_j| = 2^j$ . Pick and fix any bijection  $f$  from  $S_j$  to  $\{1, \dots, 2^j\}$ . The map  $\psi$  will be defined on

$$D = (\cup_{s=0}^j S_s) \cup \{w0, w1 : w \in S_j \text{ and } f(w) + 2^j \leq t\}.$$

For  $w \in S_j$ , define  $\psi(w) = F_{f(w)}$  if  $f(w) + 2^j > t$  and  $\psi(w) = F_{f(w)} \cup F_{f(w)+2^j}$  if  $f(w) + 2^j \leq t$ . Also, if  $f(w) + 2^j \leq t$ , define  $\psi(w0) = F_{f(w)}$  and  $\psi(w1) = F_{f(w)+2^j}$ . So far  $\psi$  is defined on  $T = S_j \cup \{w0, w1 : w \in S_j \text{ and } f(w) + 2^j \leq t\}$ . Note that  $\cup\{\psi(w) : w \in S_j\} = X$  and for each  $w \in T$   $\psi(w)$  is closed and non-empty being a union of some non-empty subcollection of  $\{F_1, \dots, F_t\}$ . The map  $\psi$  will be extended to all of  $D$  in a finite number of steps as follows. Let  $s \geq 1$  and assume  $\psi$  has been defined on  $S_s$  in such a way that  $\cup\{\psi(w) : w \in S_s\} = X$  and for each  $w \in S_s$   $\psi(w)$  is a union of some non-empty subcollection of  $\{F_1, \dots, F_t\}$ . For  $w \in S_{s-1}$ ,  $\psi(w)$  is defined to be  $\psi(w0) \cup \psi(w1)$ . Clearly this extends  $\psi$  to  $S_{s-1}$  with the same two properties described for  $S_s$ . In the last step  $s = 1$ . Since  $\{\eta\} = S_0 \subseteq D$ ,  $\psi(\eta) = X$  and so i) holds. Since  $S_1 = \{0, 1\}$ , v) holds by construction. Parts ii) and iv) are clear from the definition of  $D$ . Because words of length  $j+1$  are included in  $D$  in pairs, i.e., either both of  $w0, w1$  or neither, part vi) follows easily from the definition of  $D$ . Part vi) and the definition yield vii). Next let  $1 \leq k \leq t$ . If  $k \leq 2^j$ , there is  $v \in S_j$  with  $f(v) = k$ . For  $k + 2^j > t$ , the initial step of the definition shows that  $\psi(v) = F_k$ , and, for  $k + 2^j \leq t$ , it says  $\psi(v0) = F_k$ . If  $k > 2^j$ ,  $k - 2^j \leq t - 2^j < 2^j$  and so there is  $v \in S_j$  with  $f(v) = k - 2^j$  and by definition  $\psi(v1) = F_k$ . Thus viii) holds. For part ix), let  $w \in D$  with  $w0 \notin D$ . By the definition of  $D$  the length of  $w$  must be  $j+1$  or  $j$ . By definition of  $\psi$ , every word of length  $j+1$  maps to some  $F_i$ . If its length is  $j$ , since  $w0 \notin D$ ,  $f(w) + 2^j > t$  and so  $\psi(w) = F_{f(w)}$ . Thus ix) holds. Finally, for iii), let  $wv \in D$ . By ii),  $w \in D$ . If  $v = \eta$ ,  $wv = w$  so that  $\psi(wv) = \psi(w)$ . Now assume that iii) holds for any word of length  $k$  and suppose  $v$  is of length  $k+1$ . Then  $v = x0$  or  $v = x1$  for some  $x$  of length  $k$ . Since  $\text{length}(wx) < \text{length}(wv)$ ,  $wx \in D$  by ii). By the induction hypothesis,  $\psi(wx) \subseteq \psi(w)$ . By definition of  $\psi$ ,  $\psi(wx) = \psi(wx0) \cup \psi(wx1)$ . Because  $\psi(wv)$  is one of the two,  $\psi(wv) \subseteq \psi(w)$  as required.

Comment: See R26.Add.6 below for a different presentation of a key step in the previous proof.

**Lemma R26.15** Let  $(X, \tau)$  be a topological space and let  $X = \cup_{i=1}^t F_i$ , where each  $F_i$  is a non-empty closed subset and  $t \geq 2$  is in  $\mathbf{N}$ . Assume  $\{F_i : 1 \leq i \leq t\}$  is a pairwise disjoint collection. Let  $\psi$  be constructed as in the proof of R26.14. Assume  $u \neq v$  are in  $\text{dom}(\psi)$  with  $\text{length}(u) = \text{length}(v)$ . Then  $\psi(u) \cap \psi(v) = \emptyset$ . In addition, if  $x \neq y$  are in  $\text{dom}(\psi)$  with  $x0, y0 \notin \text{dom}(\psi)$ , then  $\psi(x) \cap \psi(y) = \emptyset$ .

Proof: Let  $j$  be the unique positive integer such that  $2^j \leq t < 2^{j+1}$  and continue with  $\psi$  and the associated notation as in the proof of R26.14. First assume  $u$  and  $v$  are in the initial part of the domain of  $\psi$ , i.e.,  $S_j \cup \{w0, w1 : w \in S_j \text{ and } f(w) + 2^j \leq t\}$ . First assume  $\text{length}(u) = j$  and note that because  $u, v \in S_j$ ,  $1 \leq f(u), f(v) \leq 2^j$ , and  $f$  is one-to-one,  $f(u), f(v), f(u) + 2^j$ , and  $f(v) + 2^j$  are four different numbers. If  $f(u) + 2^j \leq t$  and  $f(v) + 2^j \leq t$ ,  $\psi(u) = F_{f(u)} \cup F_{f(u)+2^j}$  and  $\psi(v) = F_{f(v)} \cup F_{f(v)+2^j}$ . By the assumed pairwise disjointness  $\psi(u) \cap \psi(v) = \emptyset$ . If  $f(u) + 2^j \leq t$  and  $f(v) + 2^j > t$ ,  $\psi(u) = F_{f(u)} \cup F_{f(u)+2^j}$  and  $\psi(v) = F_{f(v)}$ . For the same reasons,  $\psi(u) \cap \psi(v) = \emptyset$ . Similarly, if  $f(u) + 2^j > t$  and

$f(v)+2^j \leq t$ , the conclusion holds. If  $f(u)+2^j > t$  and  $f(v)+2^j > t$ , by the construction of  $\psi$ ,  $\psi(u) = F_{f(u)}$  and  $\psi(v) = F_{f(v)}$ . Those two sets are disjoint by hypothesis. Now assume  $\text{length}(u) = j + 1$ . Write  $u = xa$  and  $v = yb$  where  $x, y \in S_j$  and  $a, b \in \{0, 1\}$ . If  $x \neq y$ , by the first case  $\psi(x) \cap \psi(y) = \emptyset$ . Since  $\psi(u) \subseteq \psi(x)$  and  $\psi(v) \subseteq \psi(y)$ ,  $\psi(u) \cap \psi(v) = \emptyset$  as required. If  $x = y$ , by the definition of  $\psi$ ,  $\psi(u) = F_i$  and  $\psi(v) = F_l$  where  $i \neq l$ . By the pairwise disjointness assumption again,  $\psi(u) \cap \psi(v) = \emptyset$ . Now assume the conclusion holds for any two words of length  $s$ , where  $j \geq s \geq 2$ , and suppose  $\text{length}(u) = s - 1$ . By R26.14vii  $\psi(u) = \psi(u0) \cup \psi(u1)$  and  $\psi(v) = \psi(v0) \cup \psi(v1)$ . Since  $u \neq v$ ,  $u0, u1, v0, v1$  are four distinct words of length  $s$ . By the inductive hypothesis, the first conclusion holds. Finally assume  $x \neq y$  are in  $\text{dom}(\psi)$  with  $x0, y0 \notin \text{dom}(\psi)$ . By the definition of  $\text{dom}(\psi)$ ,  $x, y$  must be in  $S_j \cup \{w0, w1 : w \in S_j \text{ and } f(w) + 2^j \leq t\}$ . If both have the same length, the first conclusion applies. If one, say  $x$ , has length  $j$  and the other has length  $j + 1$ , there is  $z$  of length  $j$  such that  $y$  is either  $z0$  or  $z1$ . Because  $x0 \notin \text{dom}(\psi)$ ,  $x \neq z$ . By the first conclusion  $\psi(x) \cap \psi(z) = \emptyset$  and by R26.14iii  $\psi(y) \subseteq \psi(z)$  so that  $\psi(x) \cap \psi(y) = \emptyset$ .

**Lemma R26.16** Let  $E$  be a compact metric space. Then there is an auxiliary function  $\phi$  for  $E$  such that  $\phi(v) \neq \emptyset$  for every  $v \in M_2$  and, if  $w$  is an infinite word,  $\text{diam}(\phi(w_n)) \rightarrow 0$  as  $n \rightarrow \infty$ .

Proof: First, a sequence of maps will be defined by induction. Let  $\rho$  be a metric for  $E$  and, for  $e \in E$  and  $\epsilon > 0$ , let  $\overline{B_\epsilon(e)}$  denote the closure of the open  $\epsilon$ -ball centered at  $e$ . Pick  $m_1 \geq 1$ . By compactness applied to the open balls, there are  $e_1, \dots, e_t$  such that  $E = \cup_{i=1}^t \overline{B_{1/m_1}(e_i)}$ . Apply R26.14 to obtain  $\phi_1$  on domain  $D_1$  to the collection of unions from  $\{\overline{B_{1/m_1}(e_1)}, \dots, \overline{B_{1/m_1}(e_t)}\}$ , where  $\phi_1$  has the lemma's properties, which easily imply the 8 properties listed below. Now assume  $\phi_1, \dots, \phi_n$  with domains  $D_1 \subseteq D_2 \subseteq \dots \subseteq D_n$  have been constructed with the following properties:

- i)  $D_n \subseteq M_2$  is finite and  $\{w \in M_2 : \text{length}(w) \leq n\} \subseteq D_n$ .
- ii) For  $2 \leq i \leq n$ ,  $\phi_{i-1} = \phi_i|_{D_{i-1}}$ .
- iii)  $\phi_n(\eta) = E$ .
- iv) If  $wv \in D_n$ , then  $w \in D_n$ .
- v) For  $w \in D_n$ ,  $w0 \in D_n$  if and only if  $w1 \in D_n$ .
- vi) For  $w \in D_n$  with  $w0 \in D_n$ ,  $\phi_n(w) = \phi_n(w0) \cup \phi_n(w1)$ .
- vii) If  $w \in D_n$  and  $w0 \notin D_n$ ,  $\text{diam}(\phi_n(w)) \leq 2/n$ .
- viii) For every  $w \in D_n$ ,  $\phi_n(w)$  is a non-empty closed set.

Now pick  $m_{n+1} \geq n + 1$  and call  $w \in M_2$  a terminal word (of  $D_n$ ) if  $w \in D_n$  and  $w0 \notin D_n$ . There are finitely many terminal words. Note that no proper initial subword of a terminal word is a terminal word by properties iv) and v). For each terminal word  $w$ , by compactness  $\phi_n(w) = \cup_{i=1}^{t(w)} (\overline{B_{1/m_{n+1}}(x_i(w))} \cap \phi_n(w))$ , where  $t(w)$  in  $\mathbf{N}$  and  $x_1(w), \dots, x_{t(w)}(w)$  in  $\phi_n(w)$ . By R26.14 there is  $\psi_w$  from a finite domain  $A_w$  to the collection of unions from the family  $\{\overline{B_{1/m_{n+1}}(x_i(w))} \cap \phi_n(w) : 1 \leq i \leq t(w)\}$ , where  $\psi_w$  has the properties listed in R26.14. The finite set  $\{wv : w \text{ is a terminal word and } v \in A_w, v \neq \eta\}$  will be denoted  $C_{n+1}(w)$ . Let  $C_{n+1}$  be the union of the  $C_{n+1}(w)$  over all terminal words. Note that if  $wv = us$  where  $w, u$  are terminal words, then  $w = u$ , since neither can be a proper initial subword of the other, and so also  $v = s$ . Thus each element of  $C_{n+1}$  has a unique representation in the specified form. Finally note that the requirement  $v \neq \eta$  implies  $D_n \cap C_{n+1} = \emptyset$ . Let  $D_{n+1} = D_n \cup C_{n+1}$ . Define  $\phi_{n+1}$  on  $D_{n+1}$

by  $\phi_{n+1}(u) = \phi_n(u)$  if  $u \in D_n$  and, if  $u \in C_{n+1}$  with  $u = wv$ ,  $\phi(u) = \psi_w(v)$ . Then  $\phi_{n+1}$  is a function; clearly  $D_n \subseteq D_{n+1}$  and ii), iii), and viii) hold for  $\phi_{n+1}$ . For i), finiteness holds since each  $A_w$  is finite. Let  $w$  be a word of length  $n + 1$ . Then  $w = ux$ , where  $\text{length}(u) = n$  and  $x \in \{0, 1\}$ . Then  $u \in D_n$ . If  $w0 \in D_n$ ,  $w \in D_n \subseteq D_{n+1}$ . Otherwise  $u$  is a terminal word of  $D_n$  and, since  $\{0, 1\} \subseteq A_u$ ,  $w = ux \in C_{n+1} \subseteq D_{n+1}$ . For iv), let  $wv \in D_{n+1}$ . Since iv) holds for  $D_n$ , assume  $wv \in C_{n+1}$ . Then  $wv = us$ , where  $u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . If  $w$  is a subword of  $u$ ,  $u = wr$  and so  $w \in D_n$  by the induction hypothesis. Otherwise  $u$  must be a proper subword of  $w$ , i.e.,  $w = ur$  where  $r \neq \eta$ . Then  $wv = urv$  and so  $rv = s \in A_u$ . By R26.14ii  $r \in A_u$  and so  $w = ur \in C_{n+1} \subseteq D_{n+1}$ . For v) let  $w \in D_{n+1}$  with  $w0 \in D_{n+1}$ . If  $w0 \in D_n$ , by iv) and v) of the induction hypothesis,  $w1 \in D_n \subseteq D_{n+1}$ . If  $w0 \in C_{n+1}$ ,  $w0 = us$ , where  $u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . Now  $s = r0$  and by R26.14vi  $r1 \in A_u$ . Thus  $w1 = ur1$  is in  $C_{n+1} \subseteq D_{n+1}$ . The converse is similar. For vi) let  $w0$  (and so  $w$ ) be in  $D_{n+1}$ . If  $w0 \in D_n$ , since  $\phi_n = \phi_{n+1}|_{D_n}$ , apply vi) of the induction hypothesis for  $D_n$ . If  $w0 \in C_{n+1}$ ,  $w0 = us0$ , where  $u$  is a terminal word of  $D_n$  and  $s0 \in A_u$ . By R26.14ii  $s \in A_u$  and so  $w = us \in C_{n+1}$ . Applying the definition and R26.14vii,  $\phi_{n+1}(w) = \psi_u(s) = \psi_u(s0) \cup \psi_u(s1) = \phi_{n+1}(w0) \cup \phi_{n+1}(w1)$ . For vii) let  $w \in D_{n+1}$  with  $w0 \notin D_{n+1}$ . Note that  $w \notin D_n$  for, if so,  $w$  is a terminal word of  $D_n$  and  $0 \in A_w$  so that  $w0$  would be in  $C_{n+1} \subseteq D_{n+1}$ . Thus  $w = us$ , where  $u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . Since  $w0 = us0$  is not in  $C_{n+1}$ ,  $s0 \notin A_u$ . By R26.14ix and the choice of  $\psi_u$ ,  $\phi_{n+1}(w) = \psi_u(s) \subseteq \overline{B_{1/m_{n+1}}(x)}$  for some  $x$ . Thus  $\text{diam}(\phi_{n+1}(w)) \leq 2/m_{n+1} \leq 2/(n + 1)$ . By induction there is an infinite sequence  $\{\phi_n\}_{n=1}^\infty$  with the properties listed above. Since the domains are nested and ii) holds,  $\phi = \cup_{n=1}^\infty \phi_n$  is a function and  $\phi|_{D_n} = \phi_n$  for every  $n$ . Since i) holds, the domain of  $\phi$  is  $M_2$ . For  $w \in M_2$ , pick  $n$  with  $w0 \in D_n$ . By vi)  $\phi(w) = \phi_n(w) = \phi_n(w0) \cup \phi_n(w1) = \phi(w0) \cup \phi(w1)$ . Thus  $\phi$  is an auxiliary function for  $E$  and by viii)  $\phi(w) \neq \emptyset$  for every  $w$ . Lastly, let  $w$  be an infinite word and let  $\epsilon > 0$ . Pick  $N$  such that  $2/N < \epsilon$ . Since  $D_N$  is finite, the set  $\{k : w_k \notin D_N\}$  is non-empty and so has a smallest element  $M$ . Then  $M \geq N + 1$  by i) and  $w_{M-1} \in D_N$ . By v) and the choice of  $M$ ,  $w_{M-1}0 \notin D_N$ . By vii)  $\text{diam}(\phi(w_{M-1})) = \text{diam}(\phi_N(w_{M-1})) \leq 2/N < \epsilon$ . For  $n \geq M$ , by R26.11  $\phi(w_n) \subseteq \phi(w_{M-1})$  and so  $\text{diam}(\phi(w_n)) < \epsilon$ . Thus the limit claim holds.

**Corollary R26.17** Let  $E$  be a compact metric space. Then there is a continuous surjection  $f : \mathbf{R}_2 \rightarrow E$ .

Proof: This is immediate from R26.16 and R26.13.

In the next lemma, notation and terminology from the proof of R26.16 will continue to be used.

**Lemma R26.18** Let  $E$  be a compact metric space. Assume for every  $n$  in  $\mathbf{N}$ , for every  $\epsilon > 0$ , and for every  $e \in E$ , there exist  $m > n$  and  $x_1, \dots, x_t$  in  $B_\epsilon(e)$  such that  $t \geq 2$  and  $\overline{B_\epsilon(e)} = \cup_{i=1}^t \overline{B_{1/m}(x_i)}$  where  $\{\overline{B_{1/m}(x_i)} : 1 \leq i \leq t\}$  is a pairwise disjoint collection. Then there is an auxiliary function  $\phi$  for  $E$  constructed as in R26.16 with the property that, if  $u, v \in M_2$  with  $u \neq v$  and  $\text{length}(u) = \text{length}(v)$ , then  $\phi(u) \cap \phi(v) = \emptyset$ .

Proof: It is possible to construct a sequence  $\{\phi_n\}_{n=1}^\infty$  as in the proof of R26.16, using at each stage the pairwise disjoint collections of closed balls guaranteed by the assumption. To see this, it is sufficient to check (by induction) that each  $\phi_n$  has the following property: If  $w \in D_n$  and  $w0 \notin D_n$ , then there is  $e \in E$  and  $\epsilon > 0$  such that  $\phi_n(w) = \overline{B_\epsilon(e)}$ . This

holds for  $\phi_1$  by R26.14ix. Assume it holds for  $\phi_n$  and let  $u \in D_{n+1}$  with  $u0 \notin D_{n+1}$ . By the construction, a terminal word of  $D_n$  is not terminal in  $D_{n+1}$  and so  $u \in C_{n+1}$ , i.e.,  $u = wv$  where  $w$  is a terminal word of  $D_n$ ,  $v \neq \eta$ , and  $v$  is in the domain of  $\psi_w$ . By definition  $\phi_{n+1}(w) = \psi_w(v)$ . By the induction hypothesis  $\phi_n(w)$  is some  $\overline{B_\epsilon(e)}$ , which is a union of closed balls by this lemma's assumption. Since  $u0 \notin D_{n+1}$ ,  $v0$  is not in the domain of  $\psi_w$  and so, by R26.14ix,  $\psi_w(v)$  is one of those closed balls. With the sequence thus constructed, as in R26.16, the auxiliary function is  $\phi = \cup_{n=1}^{\infty} \phi_n$ . Since  $D_n \subseteq D_{n+1}$  for all  $n$ ,  $\phi|_{D_n} = \phi_n$  for every  $n$ , and  $M_2 = \cup_{n=1}^{\infty} D_n$ , it is sufficient to show the following by induction: for every  $n$ , if  $u \neq v$  are in  $D_n$  with  $\text{length}(u) = \text{length}(v)$  or  $u, v$  both terminal words of  $D_n$ , then  $\phi_n(u) \cap \phi_n(v) = \emptyset$ . Since  $\phi_1$  is a map constructed as in R26.14 using a pairwise disjoint union with at least two sets, by R26.15 the claim holds for  $n = 1$ . Now assume it holds for  $n$  and let  $u \neq v$  be in  $D_{n+1}$ . First assume  $\text{length}(u) = \text{length}(v)$ . If both are in  $D_n$ , apply the induction hypothesis. If both are in  $C_{n+1}$ ,  $u = xs$  and  $v = yt$  where  $x, y$  are terminal words of  $D_n$ , neither of  $s, t$  is  $\eta$ ,  $s \in \text{dom}(\psi_x)$ , and  $t$  is in  $\text{dom}(\psi_y)$ . By definition  $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$  and  $\phi_{n+1}(v) = \psi_y(t) \subseteq \phi_n(y)$ . If  $x \neq y$ , by the induction hypothesis  $\phi_n(x) \cap \phi_n(y) = \emptyset$ . If  $x = y$ ,  $s \neq t$  and  $\text{length}(s) = \text{length}(t)$ . By R26.15  $\psi_x(s) \cap \psi_x(t) = \emptyset$ . As a third case, suppose one, say  $u$ , is in  $C_{n+1}$  and the other is in  $D_n$ . Again  $u = xs$  where  $x$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $\text{dom}(\psi_x)$ ,  $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$ , and  $\phi_{n+1}(v) = \phi_n(v)$ . Since  $\text{length}(u) = \text{length}(v)$  and  $\text{length}(s) \geq 1$ ,  $v = v_1z$  where  $\text{length}(v_1) = \text{length}(x)$ . Note that  $v_1 \in D_n$  by property iv) in the proof of R26.16 and, since  $\text{length}(z) = \text{length}(s) \geq 1$ ,  $v_10 \in D_n$  by the same property. Thus  $v_1$  is not a terminal word of  $D_n$  and so  $v_1 \neq x$ . By the induction hypothesis  $\phi_n(v_1) \cap \phi_n(x) = \emptyset$  and so  $\phi_n(v_1) \cap \phi_{n+1}(u) = \emptyset$ . It follows from vi) in the proof of R26.16 that  $\phi_n(v) \subseteq \phi_n(v_1)$  and so  $\phi_{n+1}(u) \cap \phi_{n+1}(v) = \emptyset$ . Lastly assume  $u, v$  are both terminal words of  $D_{n+1}$ . By the construction of  $\phi_{n+1}$ , if  $w$  is a terminal word of  $D_n$ ,  $w0$  is in  $D_{n+1}$ , i.e.,  $w$  is not a terminal word of  $D_{n+1}$ . Thus  $u, v$  are both in  $C_{n+1}$ . As before,  $u = xs$  and  $v = yt$  where  $x, y$  are terminal words of  $D_n$ , neither of  $s, t$  is  $\eta$ ,  $s \in \text{dom}(\psi_x)$ , and  $t$  is in  $\text{dom}(\psi_y)$ . By definition  $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$  and  $\phi_{n+1}(v) = \psi_y(t) \subseteq \phi_n(y)$ . If  $x \neq y$ , by the induction hypothesis  $\phi_n(x) \cap \phi_n(y) = \emptyset$ . If  $x = y$ ,  $s \neq t$  and  $\text{length}(s) = \text{length}(t)$ . By R26.15  $\psi_x(s) \cap \psi_x(t) = \emptyset$ .

**Corollary R26.19** Let  $E$  be a compact metric space. Assume for every  $n$  in  $\mathbf{N}$ , for every  $\epsilon > 0$ , and for every  $e \in E$ , there exist  $m > n$  and  $x_1, \dots, x_t$  in  $B_\epsilon(e)$  such that  $t \geq 2$  and  $\overline{B_\epsilon(e)} = \cup_{i=1}^t \overline{B_{1/m}(x_i)}$  where  $\{B_{1/m}(x_i) : 1 \leq i \leq t\}$  is a pairwise disjoint collection. Then  $\mathbf{R}_2$  is homeomorphic to  $E$ .

*Proof:* Use the auxiliary function  $\phi$  from R26.18 to construct a continuous surjection  $f : \mathbf{R}_2 \rightarrow E$  as in R26.13. Since both spaces are compact and Hausdorff, it is sufficient to show that  $f$  is one-to-one. Let  $\mathcal{F} \neq \mathcal{G}$  be in  $\mathbf{R}_2$ . The infinite words  $d(\mathcal{F})$  and  $d(\mathcal{G})$  are different and so there is  $n$  with  $d(\mathcal{F})_n \neq d(\mathcal{G})_n$ . By R26.18  $\phi(d(\mathcal{F})_n) \cap \phi(d(\mathcal{G})_n) = \emptyset$ . Since  $f(\mathcal{F}) \in \phi(d(\mathcal{F})_n)$  and  $f(\mathcal{G}) \in \phi(d(\mathcal{G})_n)$ ,  $f(\mathcal{F}) \neq f(\mathcal{G})$ .

In [8], for  $k \geq 2$ , the metric  $d_k$  was described and shown to generate the topology of  $\mathbf{R}_k$ . In the next two results, the open balls are assumed to be generated with  $d_k$  as the underlying metric for  $\mathbf{R}_k$ .

The next lemma is implicit in [8]. Recall the following facts and notation: For  $Z$  in a normal basis  $\mathcal{Z}$ ,  $Z^\omega$  is the set of  $\mathcal{Z}$ -filters containing  $Z$ . For  $k \geq 2$  and  $x, n \in \mathbf{N}$ ,

$C_n^x(k) = \{m \in \mathbf{N} : m \equiv x \pmod{k^n}\}$ , which is in the normal basis generating  $\mathbf{N}_k$ .

**Lemma R26.20** Let  $k \geq 2$  be in  $\mathbf{N}$  and let  $\mathcal{F}$  in  $\mathbf{R}_k$ . Let  $\mathcal{F}$  have associated sequence  $\{x_n\}_{n=1}^\infty$ . Let  $1 \geq \epsilon > 0$  and let  $j$  be the unique positive integer with  $1/k^j < \epsilon \leq 1/k^{j-1}$ . Then

- i)  $B_\epsilon(\mathcal{F})$  is clopen.
- ii)  $B_\epsilon(\mathcal{F}) = B_\epsilon(\mathcal{F}) = Z^\omega \cap \mathbf{R}_k$ , where  $Z = C_j^{x_j}(k)$ .
- iii) If  $\mathcal{G}$  is in  $B_\epsilon(\mathcal{F})$ , then  $B_\epsilon(\mathcal{G}) = B_\epsilon(\mathcal{F})$ .

Proof: Part i) is an application of R20.23ii. Part ii) is verified in the proof of R20.23. For iii) let  $\mathcal{G}$  be in  $B_\epsilon(\mathcal{F})$  and suppose  $\mathcal{G}$  has associated sequence  $\{y_n\}_{n=1}^\infty$ . By ii)  $C_j^{x_j}(k) \in \mathcal{G}$  and so by R20.21  $y_j = x_j$ . Apply part ii) to  $\mathcal{G}$  in place of  $\mathcal{F}$  to obtain the conclusion.

**Lemma R26.21** Let  $k \geq 2$  be in  $\mathbf{N}$  and let  $\mathcal{F}$  in  $\mathbf{R}_k$ . Let  $\epsilon > 0$  and let  $n \in \mathbf{N}$ . Then there exist  $m > n$  and  $\mathcal{G}_1, \dots, \mathcal{G}_t$  in  $B_\epsilon(\mathcal{F})$  such that  $t \geq 2$  and  $\overline{B_\epsilon(\mathcal{F})} = \bigcup_{i=1}^t \overline{B_{1/m}(\mathcal{G}_i)}$  where  $\{\overline{B_{1/m}(\mathcal{G}_i)} : 1 \leq i \leq t\}$  is a pairwise disjoint collection.

Proof: Let  $\mathcal{F}$  have associated sequence  $\{x_i\}_{i=1}^\infty$ . Pick  $j \geq 2$  such that  $1/k^{j-1} < \epsilon$  and  $k^{j-1} > n$ . Since  $C_j^1(k), \dots, C_j^{k^j}(k)$  are the distinct classes of equivalence mod  $k^j$ , the collection  $\{(C_j^i(k))^\omega \cap \mathbf{R}_k : 1 \leq i \leq k^j\}$  is a finite cover of  $\mathbf{R}_k$  by pairwise disjoint clopen sets. By R26.20 each of those sets is a  $1/k^{j-1}$ -open ball, any element of which can be taken as its center. Thus the elements of this cover which have a non-empty intersection with  $B_\epsilon(\mathcal{F})$  can be written as  $\{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\}$ , where each  $\mathcal{G}_i \in B_\epsilon(\mathcal{F})$ . By choice of the  $\mathcal{G}_i$ ,  $B_\epsilon(\mathcal{F}) \subseteq \bigcup \{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\}$ . Let  $\mathcal{H}_i \in B_\epsilon(\mathcal{F}) \cap B_{1/k^{j-1}}(\mathcal{G}_i)$ . Since  $1/k^{j-1} < \epsilon$ , by R26.20iii (twice),  $B_{1/k^{j-1}}(\mathcal{G}_i) = B_{1/k^{j-1}}(\mathcal{H}_i) \subseteq B_\epsilon(\mathcal{H}_i) = B_\epsilon(\mathcal{F})$ . Thus  $\bigcup \{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\} \subseteq B_\epsilon(\mathcal{F})$ . Since the balls are clopen, the conclusion follows with  $m = k^{j-1}$  provided  $t \geq 2$ . That can be shown as follows: By R10.2.5i  $x_j = x_{j-1} + sk^{j-1}$ , where  $s \in \{0, 1, \dots, k-1\}$ . Pick  $r \in \{0, 1, \dots, k-1\}$  with  $r \neq s$ . Define a sequence  $\{y_i\}_{i=1}^\infty$  by  $y_i = x_i$  for  $i \leq j-1$ ,  $y_j = x_{j-1} + rk^{j-1}$ , and  $y_{i+1} = y_i + k^i$  for  $i \geq j$ . By R10.2.6 there is  $\mathcal{G}$  in  $\mathbf{R}_k$  with  $\{y_i\}_{i=1}^\infty$  as its associated sequence. Since  $y_j \neq x_j$ , the classes  $C_j^{x_j}(k)$  and  $C_j^{y_j}(k)$  are distinct. By definition of the associated sequence,  $C_j^{y_j}(k) \in \mathcal{G}$  and  $C_j^{x_j}(k) \in \mathcal{F}$  so that  $\mathcal{G}$  and  $\mathcal{F}$  are in different elements of the cover  $\{(C_j^i(k))^\omega \cap \mathbf{R}_k : 1 \leq i \leq k^j\}$ . To verify that  $t \geq 2$ , it is now sufficient to show that  $\mathcal{G} \in B_\epsilon(\mathcal{F})$ . If  $\epsilon > 1$ , this clear since  $d_k \leq 1$  so that  $B_\epsilon(\mathcal{F}) = \mathbf{R}_k$ . If  $\epsilon \leq 1$ , there is a unique  $l$  such that  $1/k^l < \epsilon \leq 1/k^{l-1}$ . Since  $1/k^{j-1} < \epsilon \leq 1/k^{l-1}$ ,  $j > l$  and so, by definition of  $\mathcal{G}$ ,  $y_l = x_l$ . Thus  $\mathcal{G}$  is in the set  $(C_l^{x_l}(k))^\omega \cap \mathbf{R}_k$ , which is  $B_\epsilon(\mathcal{F})$  by R26.20.

**Corollary R26.22** Let  $k \geq 2$  be in  $\mathbf{N}$ . Then the topological spaces  $\mathbf{R}_2$  and  $\mathbf{R}_k$  are homeomorphic.

Proof: This follows from R26.19 and R26.21.

**Corollary R26.23** Let  $k, j \geq 2$  be in  $\mathbf{N}$ . Then the topological spaces  $\mathbf{R}_k$  and  $\mathbf{R}_j$  are homeomorphic.

Proof: Transitivity and symmetry of homeomorphism: each is homeomorphic to  $\mathbf{R}_2$ .

In [3] the the following question was left unanswered: For distinct primes  $p, q$ , are the topological spaces  $\mathbf{N}_p$  and  $\mathbf{N}_q$  homeomorphic? The next few results do not answer the question but are related.

**Lemma R26.24** Let  $X$  be an infinite discrete space. Let  $(Y, f)$  and  $(Z, g)$  be  $T_2$ -compactifications of  $X$ . Assume  $h : Y \rightarrow Z$  is a homeomorphism. Then  $h[f[X]] = g[X]$ .

Proof: Let  $x \in X$ . Since  $f[X]$  is an open, discrete subspace of  $Y$ ,  $\{f(x)\}$  is open

in  $Y$  and so  $\{h(f(x))\}$  is open in  $Z$ . By density  $\{h(f(x))\} \cap g[X]$  is non-empty, i.e.  $h \circ f(x) \in g[X]$ . Thus  $h[f[X]] \subseteq g[X]$ . Similarly, using  $h^{-1}$ ,  $h[f[X]] \supseteq g[X]$ .

**Corollary R26.25** Let  $k, j \geq 2$  be in  $\mathbf{N}$ . If the topological spaces  $\mathbf{N}_k$  and  $\mathbf{N}_j$  are homeomorphic, then the topological spaces  $\mathbf{R}_k$  and  $\mathbf{R}_j$  are homeomorphic.

Proof: Let  $h : \mathbf{N}_k \rightarrow \mathbf{N}_j$  be a homeomorphism and let  $\iota_k$  and  $\iota_j$  be the embeddings of  $\mathbf{N}$  with the discrete topology into  $\mathbf{N}_k, \mathbf{N}_j$  respectively. By R26.24  $h[\iota_k[\mathbf{N}]] = \iota_j[\mathbf{N}]$ . Since  $\mathbf{R}_k = \mathbf{N}_k - \iota_k[\mathbf{N}]$  and  $\mathbf{R}_j = \mathbf{N}_j - \iota_j[\mathbf{N}]$ ,  $h$  restricted to  $\mathbf{R}_k$  maps onto  $\mathbf{R}_j$  and so is a homeomorphism from  $\mathbf{R}_k$  to  $\mathbf{R}_j$ .

In other words, R26.23 is a necessary condition for topological spaces  $\mathbf{N}_k$  and  $\mathbf{N}_j$  to be homeomorphic. The next proposition is a partial answer as to whether it is sufficient.

**Lemma R26.26** Let  $k \geq 2$  be in  $\mathbf{N}$  and let  $\iota_k, f_k$  be the embeddings of  $\mathbf{N}$  into the compactifications  $\mathbf{N}_k, \mathbf{R}_k$  respectively. Let  $\mathcal{F}$  be in  $\mathbf{R}_k$  and let  $\{x_i\}$  be a sequence in  $\mathbf{N}$  such that  $\{\iota_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{N}_k$ . Then  $\{f_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{R}_k$ .

Proof: Since  $\mathcal{F}$  is in  $\mathbf{R}_k$ , it is a non-point ultrafilter in  $\mathbf{N}_k$  and so has an associated sequence  $\{y_n\}_{n=1}^\infty$ . Recall that for each  $i$  the associated sequence of  $f_k(x_i)$  is  $\{z_n\}_{n=1}^\infty$  where  $z_n \equiv x_i \pmod{k^n}$  for each  $n$ . By R10.2.11  $\{x_i\}$  is unbounded and for every  $m$  in  $\mathbf{N}$  there is  $i_0$  such that  $i \geq i_0$  implies  $x_i \equiv y_m \pmod{k^m}$ . Then by transitivity  $i \geq i_0$  implies  $z_m \equiv y_m \pmod{k^m}$ , i.e., since  $z_m$  and  $y_m$  are both in  $\{1, 2, \dots, k^m\}$ ,  $z_m = y_m$ . By R17.2.16  $\{f_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{R}_k$ .

The previous lemma can also be proven by using the continuity of addition and the fact that  $f_k(x) = \iota_k(x) + \mathcal{O}_k$ , where  $\mathcal{O}_k$  is the additive identity in  $\mathbf{R}_k$ . The next lemma is a partial converse of R26.26.

**Lemma R26.27** Let  $k \geq 2$  be in  $\mathbf{N}$  and let  $\iota_k, f_k$  be the embeddings of  $\mathbf{N}$  into the compactifications  $\mathbf{N}_k, \mathbf{R}_k$  respectively. Let  $\mathcal{F}$  be in  $\mathbf{R}_k$  and let  $\{x_i\}$  be an unbounded sequence in  $\mathbf{N}$  such that  $\{f_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{R}_k$ . Then  $\{\iota_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{N}_k$ .

Proof: As above, since  $\mathcal{F}$  is in  $\mathbf{R}_k$ , it is a non-point ultrafilter in  $\mathbf{N}_k$  and so has an associated sequence  $\{y_n\}_{n=1}^\infty$ . Recall that for each  $i$  the associated sequence of  $f_k(x_i)$  is  $\{z_n\}_{n=1}^\infty$  where  $z_n \equiv x_i \pmod{k^n}$  for each  $n$ . Let  $m$  be in  $\mathbf{N}$ . By R17.2.16 there is  $i_0$  such that  $i \geq i_0$  implies  $z_m = y_m$ . Then  $i \geq i_0$  implies  $x_i \equiv y_m \pmod{k^m}$ . Since  $\{x_i\}$  is given to be unbounded, by R10.2.11  $\{\iota_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{N}_k$ .

**Proposition R26.28** Let  $k, j \geq 2$  be in  $\mathbf{N}$  and let  $f_k, f_j$  be the embeddings of  $\mathbf{N}$  into the compactifications  $\mathbf{R}_k, \mathbf{R}_j$  respectively. Let  $h : \mathbf{R}_k \rightarrow \mathbf{R}_j$  be a homeomorphism such that  $h[f_k[\mathbf{N}]] = f_j[\mathbf{N}]$ . Then the topological spaces  $\mathbf{N}_k$  and  $\mathbf{N}_j$  are homeomorphic.

Proof: Let  $\iota_k$  and  $\iota_j$  be the embeddings of  $\mathbf{N}$  into compactifications  $\mathbf{N}_k, \mathbf{N}_j$  respectively. Define  $H : \mathbf{N}_k \rightarrow \mathbf{N}_j$  as follows: For  $\mathcal{F}$  in  $\mathbf{R}_k$ , let  $H(\mathcal{F}) = h(\mathcal{F})$ . For  $\mathcal{F}$  in  $\mathbf{N}_k - \mathbf{R}_k$ ,  $\mathcal{F} = \iota_k(n)$  for some  $n$ . Let  $H(\mathcal{F}) = \iota_j(m)$ , where  $h(f_k(n)) = f_j(m)$ . Since  $h$  is bijective,  $h[f_k[\mathbf{N}]] = f_j[\mathbf{N}]$ , and the embeddings are one-to-one, it is easy to check that  $H$  is a one-to-one, onto function. Since the spaces are compact and Hausdorff, the continuity of  $H$  would imply that it is a homeomorphism. Let  $C$  be a closed subset of  $\mathbf{N}_j$  and decompose  $C$  into disjoint subsets  $C_1$  and  $C_2$ , where  $C_1 = \iota_j[\mathbf{N}] \cap C$  and  $C_2 = \mathbf{R}_j \cap C$ . Note that  $H^{-1}[C_2] = h^{-1}[C_2]$  is closed in  $\mathbf{R}_k$  by continuity of  $h$ . It is also closed in  $\mathbf{N}_k$  since  $\mathbf{R}_k$  is the complement of the open  $\iota_k[\mathbf{N}]$ . Thus to show that  $H^{-1}[C] = H^{-1}[C_1] \cup H^{-1}[C_2]$  is closed in  $\mathbf{N}_k$ , it is sufficient to show that the closure

of  $H^{-1}[C_1]$  is a subset of  $H^{-1}[C]$ . Let  $\mathcal{F}$  be in the closure of  $H^{-1}[C_1]$ . Since  $\mathbf{N}_k$  is metrizable and  $H^{-1}[C_1] \subseteq \iota_k[\mathbf{N}]$ , there is a sequence  $\{x_i\}$  in  $\mathbf{N}$  with  $\iota_k(x_i) \in H^{-1}[C_1]$  for every  $i$  such that  $\{\iota_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{N}_k$ . If  $\mathcal{F}$  is in  $f_k[\mathbf{N}]$ , since  $\{\mathcal{F}\}$  is open, eventually  $\iota_k(x_i) = \mathcal{F}$ , i.e.,  $\mathcal{F} \in H^{-1}[C_1] \subseteq H^{-1}[C]$ . Thus assume  $\mathcal{F}$  is in  $\mathbf{R}_k$  so that  $H(\mathcal{F}) = h(\mathcal{F})$ . For each  $i$  let  $y_i$  be the positive integer with  $h(f_k(x_i)) = f_j(y_i)$ , i.e.,  $H(\iota_k(x_i)) = \iota_j(y_i)$  by definition. By R10.2.11 the sequence  $\{x_i\}$  is unbounded and so, since  $h, f_k$ , and  $f_j$  are one-to-one,  $\{y_i\}$  is also unbounded. By R26.26  $f_k(x_i) \rightarrow \mathcal{F}$  and so by continuity  $h(f_k(x_i)) = f_j(y_i) \rightarrow h(\mathcal{F})$ . By R26.27  $\iota_j(y_i) \rightarrow h(\mathcal{F})$ , i.e.,  $H(\iota_k(x_i)) \rightarrow H(\mathcal{F})$ . Since  $C$  is closed,  $H(\mathcal{F})$  is in  $C$ , i.e.,  $\mathcal{F}$  is in  $H^{-1}[C]$  as required.

Not every homeomorphism  $h$  from  $\mathbf{R}_k$  to  $\mathbf{R}_j$  has the property assumed in R26.28: Let  $\mathcal{F}_0$  be in  $\mathbf{R}_j$  but not in  $f_j[\mathbf{Z}]$ . Since  $\mathbf{R}_j$  with addition is a topological group, the map  $t$  defined by  $t(\mathcal{F}) = \mathcal{F} + \mathcal{F}_0$  is a homeomorphism of  $\mathbf{R}_j$  onto itself. Since  $f_j$  is an additive homomorphism, at least one of  $h$  and  $t \circ h$  does not have the assumed property.

The uniformities  $\mathcal{U}_k, \mathcal{U}_j$  in the next proposition are described in detail in R10.3.8.

**Proposition R26.29** Let  $k, j \geq 2$  be in  $\mathbf{N}$  and let  $\mathcal{U}_k, \mathcal{U}_j$  be the totally bounded uniformities corresponding to  $\mathbf{N}_k$  and  $\mathbf{N}_j$  respectively. Then  $\mathbf{N}_k$  and  $\mathbf{N}_j$  are homeomorphic topological spaces if and only if there exists a permutation  $\sigma$  of the positive integers such that  $\sigma : (\mathbf{N}, \mathcal{U}_k) \rightarrow (\mathbf{N}, \mathcal{U}_j)$  is a unimorphism.

Proof: Let  $\iota_k$  and  $\iota_j$  be the embeddings of  $\mathbf{N}$  into compactifications  $\mathbf{N}_k, \mathbf{N}_j$  respectively. First assume  $h$  is a homeomorphism from  $\mathbf{N}_k$  to  $\mathbf{N}_j$ . By R26.24  $h$  maps  $\iota_k[\mathbf{N}]$  onto  $\iota_j[\mathbf{N}]$ . Both  $\iota_k$  and  $\iota_j$  are unimorphisms onto their images and, by compactness,  $h$  is also a unimorphism. Thus  $\sigma = \iota_j^{-1} \circ h \circ \iota_k$  is the required permutation and unimorphism. Conversely let a unimorphism  $\sigma$  be given. By R7.1.3 there are unique continuous maps  $P : \mathbf{N}_k \rightarrow \mathbf{N}_j$  and  $Q : \mathbf{N}_j \rightarrow \mathbf{N}_k$  such that  $\sigma = \iota_j^{-1} \circ P \circ \iota_k$  and  $\sigma^{-1} = \iota_k^{-1} \circ Q \circ \iota_j$ . Letting  $id$  be the identity on  $\mathbf{N}$ ,  $id = \iota_j^{-1} \circ P \circ Q \circ \iota_j = \iota_k^{-1} \circ Q \circ P \circ \iota_k$ , i.e.,  $P \circ Q$  is the unique continuous map from  $\mathbf{N}_j$  to  $\mathbf{N}_j$  corresponding to  $id$  and  $Q \circ P$  is the unique continuous map from  $\mathbf{N}_k$  to  $\mathbf{N}_k$  corresponding to  $id$ . By uniqueness  $P \circ Q$  is the identity on  $\mathbf{N}_j$  and  $Q \circ P$  is the identity on  $\mathbf{N}_k$ . Thus  $P$  is a homeomorphism.

This section will conclude by showing that the remnant rings are homeomorphic to the Cantor set by using an argument based on auxiliary functions. In [1] Robert derives this result for the  $p$ -adic numbers, i.e. the remnant ring  $\mathbf{R}_p$  where  $p$  is prime, by a different method and also describes several linear and Euclidean models which illustrate the fractal nature of the spaces.

To begin, let's review the construction of the Cantor set and establish notation for what follows:  $C_1$  is obtained by removing the middle third from  $[0, 1]$ , i.e.,  $C_1$  is the union  $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . Suppose  $C_n$  has been defined and is the union of  $2^n$  pairwise disjoint closed subintervals of  $[0, 1]$ . Define  $C_{n+1}$  the union of the  $2^{n+1}$  pairwise disjoint closed subintervals obtained by removing the the middle third from each of the subintervals making up  $C_n$ . Clearly  $C_{n+1} \subseteq C_n$  and by induction  $C_n$  is defined for every  $n$ . The Cantor set  $C$  is defined by  $C = \bigcap_{n=1}^{\infty} C_n$ .  $C$  is a non-empty, closed, compact subset of  $[0, 1]$ .

**Lemma R26.30** Let  $n \in \mathbf{N}$  and let  $[a, b]$  be one of the subintervals in the union making up  $C_n$ . Then  $|a - b| = 1/3^n$  and  $a, b \in C$ .

Proof: Removing the middle third of an interval leaves two closed intervals, each one-third the length of the original. This and a routine induction show that  $|a - b| = 1/3^n$ .

For the second claim, note first that, since  $C_{j+1} \subseteq C_j$  for all  $j$ ,  $a, b \in C_i$  for all  $i \leq n$ . Moreover, when the middle third is removed from an interval, the endpoints are still in the result and are again endpoints of different intervals in the union making up the result. Thus  $a, b \in C_i$  for all  $i \geq n$ . Therefore  $a$  and  $b$  are in  $C$ .

Note that removing the middle third from an interval leaves a union of two subintervals, which can naturally be described as the left and right subintervals of the result.

**Definition R26.31** The map  $\phi_C : M_2 \rightarrow \mathcal{P}(C)$  is defined inductively:  $\phi_C(\eta) = C$ . Assume  $\phi_C$  has been defined for all words of length less than  $n$  and  $w \in M_2$  has length  $n$ . Let  $w = vx$  where  $\text{length}(v) = n - 1$  and  $x \in \{0, 1\}$ . Let  $\phi_C(v0)$  be  $\phi_C(v)$  intersected with the union of the left subintervals making up  $C_n$  and  $\phi_C(v1)$  be  $\phi_C(v)$  intersected with the union of the right subintervals making up  $C_n$ .

The diameter in the next lemma refers of course to the absolute value metric on  $[0, 1]$ .

**Lemma R26.32** The map  $\phi_C$  has the following properties.

- i) For every  $w \in M_2$ ,  $\phi_C(w)$  is a closed subset of  $C$ .
- ii) If  $w \in M_2$  has length  $n \geq 1$ ,  $\phi_C(w) = [a, b] \cap C$ , where  $[a, b]$  is one of the subintervals in the union making up  $C_n$ .
- iii) If  $w \in M_2$  has length  $n \geq 1$ ,  $\phi_C(w)$  is non-empty.
- iv) If  $w \in M_2$  has length  $n \geq 1$ ,  $\text{diam}(\phi_C(w)) = 1/3^n$ .
- v) For every  $w \in M_2$ ,  $\phi_C(w) = \phi_C(w0) \cup \phi_C(w1)$ .

Proof: Part i) follows by induction since  $C$  is closed and there are  $2^{n-1}$  closed left (or right) subintervals in the union making up  $C_n$ . For part ii),  $\phi_C(0) = [0, \frac{1}{3}] \cap C$  and  $\phi_C(1) = [\frac{2}{3}, 1] \cap C$ . Since  $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ , the claim holds for words of length one. Assume it holds for all words of length  $n$  and let  $w$  be of length  $n + 1$ . Then  $w = vx$  where  $v$  is of length  $n$  and  $x \in \{0, 1\}$ . By the induction hypothesis  $\phi_C(v) = [a, b] \cap C$ , where  $[a, b]$  is one of the subintervals making up  $C_n$ . Among the intervals making up  $C_{n+1}$ , only the left and right thirds of  $[a, b]$  have a possibly non-empty intersection with  $\phi_C(v)$ . By definition of  $\phi_C$  and the induction hypothesis,  $\phi_C(w)$  equals  $C$  intersected with either the left or right subinterval of  $[a, b]$ , both of which are subintervals in the union making up  $C_{n+1}$ . Part iii) is immediate from R26.30 and part ii). For part iv), let  $w$  have length  $n \geq 1$  and let  $\phi_C(w) = [a, b] \cap C$  where  $[a, b]$  is one of the subintervals in the union making up  $C_n$ . By R26.30  $a, b \in \phi_C(w)$  and so  $\text{diam}(\phi_C(w)) \geq |a - b| = 1/3^n$ . Since  $\phi_C(w) \subseteq [a, b]$ ,  $\text{diam}(\phi_C(w)) \leq 1/3^n$ . For v), given  $w \in M_2$  of length  $n$ , since  $C_{n+1}$  is the union of its left and right subintervals, by definition  $\phi_C(w0) \cup \phi_C(w1) = C_{n+1} \cap \phi_C(w)$ . Since  $\phi_C(w) \subseteq C \subseteq C_{n+1}$ , the desired equality holds.

**Proposition R26.33** The topological space  $\mathbf{R}_2$  is homeomorphic to the Cantor set.

Proof: By definition  $\phi_C(\eta) = C$  and so by i) and v) of the previous lemma  $\phi_C$  is an auxiliary function for  $C$ . Part iv) of the lemma shows that, if  $w$  is an infinite word,  $\text{diam}(w_n) \rightarrow 0$  as  $n \rightarrow \infty$ . That and part iii) of the allow an application of R26.13: There is a continuous, onto map  $f : \mathbf{R}_2 \rightarrow C$ . Since the spaces are compact and  $T_2$ , it is sufficient to show that the  $f$  constructed as in the proof of R26.13 is one-to-one. Let  $\mathcal{F} \neq \mathcal{G}$  be in  $\mathbf{R}_2$  with corresponding infinite words  $d(\mathcal{F}) = w$  and  $d(\mathcal{G}) = v$ . By R26.6  $w \neq v$  and so there is  $n$  with  $w_n \neq v_n$ . Assume  $n$  is the least such positive integer. Without loss of generality assume  $w_n = x0$  and  $v_n = x1$ , where  $x$  (possibly  $\eta$ ) has length  $n - 1$ . By definition  $\phi_C(w_n)$  is contained in the union of the left subintervals making up  $C_n$  and  $\phi_C(v_n)$  is contained

in the union of the right subintervals making up  $C_n$ . Since these two unions are disjoint,  $\phi_C(w_n) \cap \phi_C(v_n) = \emptyset$ . By the construction of  $f$  in the proof of R26.13,  $f(\mathcal{F}) \in \phi_C(w_n)$  and  $f(\mathcal{G}) \in \phi_C(v_n)$ . Thus  $f(\mathcal{F}) \neq f(\mathcal{G})$ .

**Corollary R26.34** Let  $k \geq 2$  be in  $\mathbf{N}$ . The topological space  $\mathbf{R}_k$  is homeomorphic to the Cantor set.

Proof: This follows from R26.23 and the symmetry and transitivity of homeomorphism.

Comment: Robert [1] has an exercise (14 p.65), which asks the reader to show that a compact metric space  $E$  is path-connected if and only there is a continuous surjection  $f : [0, 1] \rightarrow E$ . In outline, the difficult half of this can be done as follows: Use R26.17 and R26.33 to obtain a continuous surjection  $f_0 : C \rightarrow E$ . Extend  $f_0$  inductively by using appropriate curves on the closures of the removed middle thirds. For example, obtain  $f_1$  on  $C \cup [\frac{1}{3}, \frac{2}{3}]$  by pasting together  $f_0$  and a continuous curve  $\gamma : [\frac{1}{3}, \frac{2}{3}] \rightarrow E$  such that  $\gamma(\frac{1}{3}) = f_0(\frac{1}{3})$  and  $\gamma(\frac{2}{3}) = f_0(\frac{2}{3})$ . Given  $f_n$ , extend it to  $f_{n+1}$  in a similar way with finitely many curves. The desired map  $f$  is  $\cup_{n=0}^{\infty} f_n$ . This result shows the existence of a continuous surjection from  $[0, 1]$  onto  $[0, 1] \times [0, 1]$ , i.e., a space-filling curve.

Albert J. Klein 2016

<http://www.susanjkleinart.com/compactification/>

## References

1. Robert, Alain M., A Course in p-adic Analysis, Springer Verlag, New York, 2000.
2. This website, R7: Uniform Continuity and Extensions of Maps
3. This website, R10: Some Metric Compactifications of the Natural Numbers
4. This website, R12: Extension of Arithmetic Operations
5. This website, R13: Mixed Suprema
6. This website, R16: The Remnant Rings as Compactifications
7. This website, R17: Algebraic Structure of the Remnant Rings
8. This website, R20: p-adic Tools for the Remnant Rings

## Added 2019

The main result of this added subsection is that  $(\mathbf{R}_k, <_k)$  and  $(\mathbf{R}_l, <_l)$  are order isomorphic for  $k, l \geq 2$  in  $\mathbf{N}$ . It follows that the topological spaces  $\mathbf{N}_k$  and  $\mathbf{N}_l$  are homeomorphic.

The argument for R26.14 uses an arbitrary bijection from  $S_j$  to  $\{1, \dots, 2^j\}$  in the initialization of the preliminary function  $\psi$ . This added subsection in effect makes a more specific choice in the case of the remnant rings, a choice which eventually leads to a homeomorphism also preserving order. The notation for  $\epsilon$ -balls will be as above, with  $d_k$  as the underlying metric on  $\mathbf{R}_k$ . By R20.23.ii all  $d_k$ -balls are clopen.

The first lemma is a variation of R26.21.

**Lemma R26.Add.1** Let  $k, l \in \mathbf{N}$  with  $k \geq 2$ . Then

- i)  $\mathbf{R}_k = \cup_{t=1}^k B_1(f_k(t))$ .
- ii) If  $1 \leq x \leq k^l$  with  $x \in \mathbf{N}$ ,  $B_{1/k^{l-1}}(f_k(x)) = \cup_{s=0}^{k-1} B_{1/k^l}(f_k(x + sk^l))$ .
- iii) The balls in either union form a pairwise disjoint collection.

Proof: For any integer  $t$  with  $1 \leq t \leq k$ , by R26.20.ii  $B_1(f_k(t)) = (C_1^t(k))^\omega \cap \mathbf{R}_k$ . Since  $\mathbf{N} = \cup_{t=1}^k C_1^t(k)$ , by definition of the associated sequence and R20.21, every element of  $\mathbf{R}_k$  is in some  $(C_1^t(k))^\omega \cap \mathbf{R}_k$  and so i) holds. For integers  $t \neq r$  with  $1 \leq t, r \leq k$  the equivalence classes are disjoint and so  $\{B_1(f_k(t)) : 1 \leq t \leq k\}$  is a pairwise disjoint collection as claimed in iii). Now let  $x$  be an integer with  $1 \leq x \leq k^l$ . By R26.20ii again,  $B_{1/k^{l-1}}(f_k(x)) = (C_l^x(k))^\omega \cap \mathbf{R}_k$  and, for  $s \in \{0, \dots, k-1\}$ ,  $B_{1/k^l}(f_k(x + sk^l)) = (C_{l+1}^{x+sk^l}(k))^\omega \cap \mathbf{R}_k$ . Since  $C_l^x(k) = \cup_{s=0}^{k-1} C_{l+1}^{x+sk^l}(k)$ , by definition of the associated sequence and R20.21, every element of  $(C_l^x(k))^\omega \cap \mathbf{R}_k$  is in some  $(C_{l+1}^{x+sk^l}(k))^\omega \cap \mathbf{R}_k$ . Similarly  $(C_{l+1}^{x+sk^l}(k))^\omega \cap \mathbf{R}_k$  is a subset of  $(C_l^x(k))^\omega \cap \mathbf{R}_k$ . Thus ii) holds. Lastly, because the equivalence classes are pairwise disjoint, so are the corresponding balls.

**Lemma R26.Add.2** Let  $k, l, a, b \in \mathbf{N}$  with  $k \geq 2$  and  $a \not\equiv b \pmod{k^l}$ . Assume  $f_k(a) <_k f_k(b)$ . Let  $\mathcal{F} \in B_{1/k^{l-1}}(f_k(a))$  and  $\mathcal{G} \in B_{1/k^{l-1}}(f_k(b))$ . Then  $\mathcal{F} <_k \mathcal{G}$ .

Proof: Let  $\mathcal{F}$  be associated with  $\{y_n\}_{n=1}^\infty$ ,  $\mathcal{G}$  with  $\{z_n\}_{n=1}^\infty$ ,  $f_k(a)$  with  $\{a_n\}_{n=1}^\infty$ , and  $f_k(b)$  with  $\{b_n\}_{n=1}^\infty$ . Since  $a_l \equiv a \pmod{k^l}$ ,  $C_l^{a_l}(k) = C_l^a(k)$ . By R26.20ii  $B_{1/k^{l-1}}(f_k(a)) = (C_l^a(k))^\omega \cap \mathbf{R}_k$ . Since  $\mathcal{F}$  is in the ball,  $C_l^a(k) \in \mathcal{F}$  and, by the definition of the associated sequence,  $C_l^{y_l}(k) \in \mathcal{F}$ . Since two elements of a  $\mathcal{Z}_k$ -filter must have a non-empty intersection,  $C_l^{y_l}(k) = C_l^a(k)$ , i.e.,  $y_l \equiv a \equiv a_l \pmod{k^l}$ . Since  $1 \leq y_l, a_l \leq k^l$ ,  $y_l = a_l$ . By R26.7  $y_n = a_n$  for  $1 \leq n \leq l$ . Similarly,  $z_l \equiv b \pmod{k^l}$  and  $z_n = b_n$  for  $1 \leq n \leq l$ . Also note that  $f_k(a) <_k f_k(b)$  implies  $f_k(a) \neq f_k(b)$ . Since  $f_k$  is a function,  $a \neq b$ . Now let  $M$  be the smallest of  $\{n : a_n \neq b_n\}$ . Since  $a \not\equiv b \pmod{k^l}$ ,  $C_l^a(k) \cap C_l^b(k) = \emptyset$  and so  $a_l \neq b_l$ . Thus  $M \leq l$  so that  $M$  is also the smallest of  $\{n : y_n \neq z_n\}$ . By hypothesis and R19.Add.2,  $a_M < b_M$ . Thus  $y_M < z_M$ . By R19.Add.2 again  $\mathcal{F} <_k \mathcal{G}$ .

For convenience the usage of  $<_k$  will be extended as follows: for  $S, T \subseteq \mathbf{R}_k$  such that  $\mathcal{F} \in S$  and  $\mathcal{G} \in T$  imply  $\mathcal{F} <_k \mathcal{G}$ ,  $S <_k T$  will be written. Of course, a pair of subsets may not be related, but related subsets must be disjoint. With this convention, the conclusion of R26.Add.2 could be written  $B_{1/k^{l-1}}(f_k(a)) <_k B_{1/k^{l-1}}(f_k(b))$ .

It will also be convenient to write  $\mathcal{M}$  for the set of all infinite words with characters from  $\{0, 1\}$ . With the natural order on  $\{0, 1\}$ , the dictionary order  $<_D$  is a linear order on  $\mathcal{M}$ . As in the proof of R26.14, for  $t \in \mathbf{N}$ ,  $S_t = \{w \in \mathcal{M} : \text{length}(w) = t\}$ . Dictionary order applies to the pairs from  $S_t$  and the same notation will be used. The  $<_D$ -ordered indexing of  $S_t$  is  $\{u_i : 1 \leq i \leq 2^t\}$ , where  $u_i \in S_t$  for all  $i$  and  $u_i <_D u_{i+1}$  for  $1 \leq i \leq 2^t - 1$ .

Similarly, for  $k \geq 2$  the  $k$ -ordered indexing or  $k$ -ordering of a non-empty, finite subset  $S$  of  $\mathbf{N}$  will be written  $\{x_i : 1 \leq i \leq |S|\}$ , where  $x_i \in S$  for all  $i$  and  $f_k(x_i) <_k f_k(x_{i+1})$  for  $1 \leq i \leq |S| - 1$ . By R26.Add.2, if the elements of  $S$  represent distinct equivalence classes mod  $k^l$ , the  $k$ -ordered indexing of  $S$  induces the  $k$ -ordered indexing of the corresponding  $1/k^{l-1}$ -balls.

The next lemma is used repeatedly, sometimes implicitly, in the following definition and related results.

**Lemma R26.Add.3** Let  $k, j \in \mathbf{N}$  with  $2^j \leq k < 2^{j+1}$ . Let  $c = 2^{j+1} - k$ . Then  $c \leq 2^j$  and  $k - c$  is even. Moreover,  $\{i : c < i \leq 2^j\} = \{c + t : 1 \leq t \leq (k - c)/2\}$ . Lastly, the map  $t \mapsto \{c + 2t - 1, c + 2t\}$  from the domain  $\{n \in \mathbf{N} : 1 \leq n \leq (k - c)/2\}$  has disjoint images for  $s \neq t$  and the union of the images is  $\{n \in \mathbf{N} : c + 1 \leq n \leq k\}$ .

Proof: Because  $-k \leq -2^j$ ,  $c = 2^{j+1} - k \leq 2^{j+1} - 2^j = 2^j$ . Next  $k - c$  equals  $k - (2^{j+1} - k) = 2(k - 2^j)$ , which is even. The second claim follows easily from the observation that  $2^j - c = k - 2^j$ , which is  $(k - c)/2$ . Next, for the map with  $s \neq t$ , the odd-determined elements of the images ( $c + 2t - 1$  and  $c + 2s - 1$ ) are distinct, as are the even-determined elements. It follows easily that the images of  $s$  and  $t$  are disjoint. Lastly let  $n \in \mathbf{N}$  with  $c + 1 \leq n \leq k$ . If  $n - c$  is even, the integer  $t = (n - c)/2$  is in the specified domain and  $n = c + 2t$  is in the image of  $t$ . If  $n - c$  is odd,  $t = (n - c + 1)/2$  is an integer between 1 and  $(k - c)/2 + 1/2$ , i.e.,  $t$  is in the specified domain. Clearly,  $n = c + 2t - 1$  is in the image of  $t$ .

**Definition R26.Add.4** Let  $k, l, x$  be integers with  $k \geq 2$ ,  $x \geq 1$ , and  $l \geq 0$ . Let  $X$  be the (clopen in  $\mathbf{R}_k$ ) set  $\cup_{s=0}^{k-1} B_{1/k^l}(f_k(x + sk^l))$ . Assume  $j \in \mathbf{N}$  is such that  $2^j \leq k < 2^{j+1}$ . Let  $\{x_i : 1 \leq i \leq k\}$  be the  $k$ -ordering of  $S = \{x + sk^l : 0 \leq s \leq k - 1\}$  and  $\{u_i : 1 \leq i \leq 2^j\}$  be the  $<_D$ -indexed ordering of  $S_j$ . Define  $g_{lx} : S_j \rightarrow \mathcal{P}(X)$  as follows: For  $1 \leq i \leq 2^{j+1} - k$ , let  $g_{lx}(u_i) = B_{1/k^l}(f_k(x_i))$ . For  $c = 2^{j+1} - k$  and  $t \in \{1, \dots, (k - c)/2\}$ ,  $g_{lx}(u_{c+t})$  is defined to be  $B_{1/k^l}(f_k(x_{c+2t-1})) \cup B_{1/k^l}(f_k(x_{c+2t}))$ .

**Lemma R26.Add.5** Let  $k, l, x$  be integers with  $k \geq 2$ ,  $x \geq 1$ , and  $l \geq 0$ . Let  $S = \{x + sk^l : 0 \leq s \leq k - 1\}$ . Assume  $j$  is the integer with  $2^j \leq k < 2^{j+1}$ . Let  $\{x_i : 1 \leq i \leq k\}$  be the  $k$ -ordering of  $S$  and  $\{u_i : 1 \leq i \leq 2^l\}$  be the  $<_D$ -indexed ordering of  $S_j$ . Then

- i) If  $x_i \in S$ , then there is a unique  $u_n \in S_j$  such that  $B_{1/k^l}(f_k(x_i)) \subseteq g_{lx}(u_n)$ .
- ii) If  $u_i, u_n \in S_j$  with  $u_i <_D u_n$ , then  $g_{lx}(u_i) <_k g_{lx}(u_n)$ .

Proof: First note that for  $a, b \in \mathbf{N}$  with  $a \not\equiv b \pmod{k^{l+1}}$ , the balls  $B_{1/k^l}(f_k(a))$  and  $B_{1/k^l}(f_k(b))$  are disjoint by R26.20.ii, because  $C_{l+1}^a(k) \cap C_{l+1}^b(k) = \emptyset$ . Thus, because no two elements of  $S$  are equivalent mod  $k^{l+1}$ , the balls used to assign the values of  $g_{lx}$  are pairwise disjoint. Now let  $x_i \in S$ . By the disjointness of the balls and R26.Add.3, there is at most one  $u_n \in S_j$  such that  $B_{1/k^l}(f_k(x_i)) \subseteq g_{lx}(u_n)$ . If  $1 \leq i \leq 2^{j+1} - k$ , by definition  $g_{lx}(u_i) = B_{1/k^l}(f_k(x_i))$ . If  $2^{j+1} - k < i \leq k$ , let  $c = 2^{j+1} - k$ . If  $i - c$  is even, let  $t = (i - c)/2$ . Clearly,  $t$  is an integer and  $i - c \geq 2$  so that  $t \geq 1$ . Also  $i - c \leq k - c$  so that  $t \leq (k - c)/2$ . Clearly,  $i = c + 2t$  and so by definition  $B_{1/k^l}(f_k(x_i)) \subseteq g_{lx}(u_{c+t})$ . Next suppose  $i - c$  is odd. Let  $t = (i - c + 1)/2$ . Clearly  $t$  is a positive integer. Because  $i = k$  would imply  $i - c = k - c$  which is even, in this case  $i \leq k - 1$  so that  $i - c + 1 \leq k - c$ , i.e.,  $t \leq (k - c)/2$ . Now  $i = c + 2t - 1$  and so by definition  $B_{1/k^l}(f_k(x_i)) \subseteq g_{lx}(u_{c+t})$ . Thus the first claim holds. Now suppose  $u_i, u_n \in S_j$  with  $u_i <_D u_n$ . Because a  $<_D$ -indexed ordering is assumed,  $i < n$ . Again let  $c = 2^{j+1} - k$ . If  $i, n \leq c$ , since  $k$ -ordering is given,  $f_k(x_i) <_k f_k(x_n)$  and so by R26Add.2 the balls are related in the same way, i.e.,  $g_{lx}(u_i) <_k g_{lx}(u_n)$ . If  $i \leq c < n$ , the image of  $u_n$  is the union of two balls determined by subscripts greater than  $c > i$ . The conclusion follows easily from R26Add.2. Lastly assume  $c < i < n$ . Write  $i = c + s$  and  $n = c + t$  with  $s, t \in \mathbf{N}$  and  $s < t$  so that  $2s < 2t$ . Also  $2s < 2t - 1$  because of the even/odd pairing. Then  $g_{lx}(u_i) <_k g_{lx}(u_n)$  by the definition of  $g_{lx}$  and a routine application of R26.Add.2.

The following lemma is a better version of a key step in the proof of R26.14.

**Lemma R26.Add.6** Let the set  $T = \cup_{i=1}^t T_i$  and let  $j \in \mathbf{N}$ . Assume  $\psi_1 : S_j \rightarrow \mathcal{P}(T)$  is such that each word in  $S_j$  maps to a union of one or more elements of  $\{T_i : 1 \leq i \leq t\}$  and  $\cup\{\psi_1(w) : w \in S_j\} = T$ . Let  $E_j = \{w \in M_2 : \text{length}(w) \leq j\}$ . Then there is a unique  $\Psi : E_j \rightarrow \mathcal{P}(T)$  such that

- i)  $\Psi(\eta) = T$ .
- ii) For each  $w \in E_j$   $\Psi(w)$  is a union of one or more elements of  $\{T_i : 1 \leq i \leq t\}$ .
- iii) For  $w$  with  $w0, w1 \in E_j$   $\Psi(w) = \Psi(w0) \cup \Psi(w1)$ .
- iv) For  $w \in S_j$   $\Psi(w) = \psi_1(w)$ .

Proof: Proceed by induction on  $j$ . For  $j = 1$ ,  $E_1 = \{\eta, 0, 1\}$  and  $\Psi$  must be the extension of  $\psi_1$  with  $\Psi(\eta) = \psi_1(0) \cup \psi_1(1)$ , which is  $T$  by hypothesis. Clearly  $\Psi$  works and is unique. Now assume the conclusion holds for  $j$  and suppose  $\psi_1$  is defined on  $S_{j+1}$ . Let  $\psi_2$  be defined on  $S_j$  by  $\psi_2(w) = \psi_1(w0) \cup \psi_1(w1)$ . The induction hypothesis applied to  $\psi_2$  yields an extension  $\sigma$  of  $\psi_2$ . Define  $\Psi$  on  $E_{j+1}$  by  $\Psi(w) = \sigma(w)$  if  $\text{length}(w) \leq j$  and  $\Psi(w) = \psi_1(w)$  if  $w \in S_{j+1}$ . Clearly  $\Psi$  has the required properties. If  $\bar{\Psi}$  is another such,  $\bar{\Psi}$  restricted to the set of words of length less than or equal to  $j$  is a suitable extension of  $\psi_2$  and so must be  $\sigma$ . It follows easily that  $\bar{\Psi} = \Psi$ .

**Corollary R26.Add.7** Let  $k, j, l, x$  be integers with  $k \geq 2$ ,  $x \geq 1$ ,  $l \geq 0$ , and  $2^j \leq k < 2^{j+1}$ . Let  $S = \{x + sk^l : 0 \leq s \leq k - 1\}$  and  $X = \cup\{B_{1/k^l}(f_k(y)) : y \in S\}$ . Let  $\{x_i : 1 \leq i \leq k\}$  be the  $k$ -ordering of  $S$  and  $\{u_n : 1 \leq n \leq 2^j\}$  be the  $<_D$ -indexed ordering of  $S_j$ . Let  $E_j = \{w \in M_2 : \text{length}(w) \leq j\}$ . Let  $G$  be the extension of  $g_{lx}$  to  $E_j$  guaranteed by the previous lemma. If  $u, v \in E_j$  are of equal length and  $u <_D v$ , then  $G(u) <_k G(v)$ .

Proof: Let  $u, v \in E_j$  be of equal length with  $u <_D v$ . If their length is  $j$ , the conclusion follows from R26.Add.5ii, since  $G$  is an extension of  $g_{lx}$ . Let  $\text{length}(u) = t$  where  $t \leq j - 1$  and the conclusion hold for words of length  $t + 1$ . The words  $u0 <_D u1 <_D v0 <_D v1$  are of length  $t + 1$  and so  $G(u0) <_k G(u1) <_k G(v0) <_k G(v1)$ . Since  $G(u) = G(u0) \cup G(u1)$  and  $G(v) = G(v0) \cup G(v1)$ , it is routine to check that  $G(u) <_k G(v)$  as required.

The next lemma is a version of R26.14 with an added property. In brief, the arbitrary bijection  $f$  used in the proof of R26.14 is replaced by the  $<_D$ -indexed ordering of  $S_j$  and a suitable  $g_{lx}$  is used as the initial map. The complete proof given here is somewhat repetitive but also uses a partly different presentation.

**Lemma R26.Add.8** Let  $k, l, x$  be integers with  $k \geq 2$ ,  $x \geq 1$ , and  $l \geq 0$ . Let  $S = \{x + sk^l : 0 \leq s \leq k - 1\}$  and  $X = \cup\{B_{1/k^l}(f_k(y)) : y \in S\}$ . Then there is a finite set  $E \subseteq M_2$  and  $\psi$  from  $E$  to the non-empty closed subsets of  $X$  such that

- i)  $\eta \in E$  and  $\psi(\eta) = X$ .
- ii) If  $w \in E$  and  $v \in M_2$  with  $\text{length}(v) < \text{length}(w)$ , then  $v \in E$ .
- iii) If  $wv \in E$ , then  $\psi(wv) \subseteq \psi(w)$ .
- iv) There is  $j \geq 1$  such that  $\{w \in M_2 : \text{length}(w) \leq j\} \subseteq E$ .
- v)  $0, 1 \in E$  and  $\psi(0) \cup \psi(1) = X$ .
- vi) If  $w \in E$ , then  $w0 \in E$  if and only if  $w1 \in E$ .
- vii) If  $w$  and  $w0$  are in  $E$ , then  $\psi(w) = \psi(w0) \cup \psi(w1)$ .
- viii) For  $y \in S$  there is  $w \in E$  such that  $\psi(w) = B_{1/k^l}(f_k(y))$  and  $w0 \notin E$ .
- ix) If  $w \in E$  and  $w0 \notin E$ , then  $\psi(w) = B_{1/k^l}(f_k(y))$  for some  $y \in S$ .
- x) If  $u, w$  are equal-length elements of  $E$  with  $u <_D w$ , then  $\psi(u) <_k \psi(w)$ .

Proof: Assume  $j$  is the unique positive integer with  $2^j \leq k < 2^{j+1}$  and let  $E_j$  be the set  $\{w \in M_2 : \text{length}(w) \leq j\}$ . Let  $\{x_i : 1 \leq i \leq k\}$  be the  $k$ -ordering of  $S$  and  $\{u_i : 1 \leq i \leq 2^j\}$  be the  $<_D$ -indexed ordering of  $S_j$ . Let  $G$  be the unique extension of  $g_{lx}$  established in R26.Add.6. The domain of  $G$  is  $E_j$ . Next  $\psi$  will be defined. Let  $c = 2^{j+1} - k$  and  $E^* = \{u_i0, u_i1 : c < i \leq 2^j\}$ . By R26.Add.3  $E^*$  can also be represented as  $\{u_{c+t}0, u_{c+t}1 : 1 \leq t \leq (k-c)/2\}$ . Note that the elements of  $E^*$  have length  $j+1$  and so  $\psi$  can be defined on  $E = E_j \cup E^*$  by  $\psi = G$  on  $E_j$ ,  $\psi(u_{c+t}0) = B_{1/k^{m-1}}(f_k(x_{c+2t-1}))$  and  $\psi(u_{c+t}1) = B_{1/k^{m-1}}(f_k(x_{c+2t}))$ . Property x) will be verified first. Let  $u, w$  be equal-length elements of  $E$  with  $u <_D w$ . Either both are in  $E_j$  or both are in  $E^*$ , since elements of  $E^*$  are all of length  $j+1$ . If both are in  $E_j$ , the conclusion follows from R26.Add.7. Assume both are in  $E^*$ . As a first case, suppose both have the same initial string. Since  $u <_D w$ , it must be that  $u = u_{c+t}0$  and  $w = u_{c+t}1$ . Then  $\psi(u) = B_{1/k^l}(f_k(x_{c+2t-1}))$  and  $\psi(w) = B_{1/k^l}(f_k(x_{c+2t}))$ . Because the  $k$ -ordering of  $S$  is assumed,  $\psi(u) <_k \psi(w)$ . In the other case, the initial strings must be different, say  $u = u_{c+s}a$  and  $w = u_{c+t}b$ , where  $s \neq t$  and  $a, b \in \{0, 1\}$ . Since the initial strings are different, the dictionary order of  $u$  and  $w$  is determined by their order, i.e.,  $u_{c+s} <_D u_{c+t}$ . Because the  $<_D$ -indexed ordering of  $S_j$  is assumed,  $c+s < c+t$ . Moreover,  $c+2s \leq c+2t-1$  since  $c+2s < c+2t$ . Equality cannot hold because it would imply  $2s = 2t-1$ . Thus  $c+2s < c+2t-1$ . Because the  $k$ -ordering of  $S$  is assumed,  $\psi(u) <_k \psi(w)$ . Property i) holds for  $G$  (and so for  $\psi$ ) by R26.Add.6i. Property ii) holds since the shorter word must be in  $E_j$ .  $E_j \subseteq E$  verifies iv). Property vi) holds for  $w$  of length less than or equal to  $j-1$ . For elements of  $S_j$ , say  $u_i$  with  $1 \leq i \leq c$ , neither  $u_i0$  nor  $u_i1$  is in  $E$ . For  $u_i$  with  $c < i \leq 2^j$ ,  $\{u_i0, u_i1\} \subseteq E^*$  by definition. Lastly  $w \in E^*$  implies neither  $w0$  nor  $w1$  is in  $E$ . Thus vi) holds. Property vii) holds for any word  $w$  with  $\text{length}(w) \leq j-1$  because it holds for  $G$  (and so for  $\psi$ ) by R26.Add.6iii. It is vacuously true on  $E^*$  and  $\{u_i : 1 \leq i \leq c\}$ . For  $u_i$  with  $c < i \leq 2^j$ , vii) holds by the definitions of  $\psi$  and  $g_{lx}$ , which  $G$  extends. Properties vii) and i) imply v). For iii) assume  $wv \in E$  and proceed by induction: If  $\text{length}(v) = 0$ , i.e.  $v = \eta$ , clearly  $\psi(wv) \subseteq \psi(w)$ . Now assume iii) holds for any  $v$  of length less than or equal  $t$  and suppose  $\text{length}(v) = t+1$ . Let  $v = ux$  where  $x \in \{0, 1\}$ . By ii)  $wu \in E$  and by vi)  $wu0$  and  $wu1$ , one of which is  $wv$ , are both in  $E$ . By vii)  $\psi(wu) = \psi(wu0) \cup \psi(wu1)$  and by the induction hypothesis  $\psi(wu) \subseteq \psi(w)$ . It follows that  $\psi(wv) \subseteq \psi(w)$  as required. For viii) let  $x_i \in S$ . If  $1 \leq i \leq c$ , by definition  $\psi(u_i) = G(u_i) = g_{lx}(u_i) = B_{1/k^l}(f_k(x_i))$  and  $u_i0 \notin E$ . If  $c+1 \leq i \leq k$ , by R26.Add.3 there is  $t$  with  $1 \leq t \leq (k-c)/2$  such that  $i \in \{c+2t-1, c+2t\}$ . For  $w = u_{c+t}0$  if  $i = c+2t-1$  and  $w = u_{c+t}1$  if  $i = c+2t$ ,  $\psi(w) = B_{1/k^l}(f_k(x_i))$ . In either case  $w0$  has length  $j+2$  and so  $w0 \notin E$ . Lastly, ix) holds: If  $w \in E$  with  $w0 \notin E$ , either  $w \in \{u_i : 1 \leq i \leq c\}$  or  $w = u_{c+t}x$  where  $x \in \{0, 1\}$  and  $1 \leq t \leq (k-c)/2$ . By definition of  $\psi$ , those are the elements which map to the balls.

**Lemma R26.Add.9** Let  $k, l, x$  be integers with  $k \geq 2$ ,  $x \geq 1$ , and  $l \geq 0$ . Let  $S = \{x + sk^l : 0 \leq s \leq k-1\}$  and  $X = \cup\{B_{1/k^l}(f_k(y)) : y \in S\}$ . Let  $\psi$  with domain  $E$  be constructed as in the previous lemma. Assume  $u \neq v$  are in  $E$  with  $\text{length}(u) = \text{length}(v)$ . Then  $\psi(u) \cap \psi(v) = \emptyset$ . In addition, if  $x \neq y$  are in  $E$  with  $x0, y0 \notin E$ , then  $\psi(x) \cap \psi(y) = \emptyset$ .

Proof: Let  $j$  be the unique positive integer such that  $2^j \leq k < 2^{j+1}$  and use the notation defining  $\psi$  from the proof of R26.Add.8. Because no two elements of  $S$  are

equivalent class mod  $k^{l+1}$ , the set of balls  $\{B_{1/k^l}(f_k(y)) : y \in S\}$  is a pairwise disjoint collection. By R26.Add.3 the images of elements in  $E^*$  are distinct, and so disjoint, balls. In addition, the elements  $e \in E$  with  $e0 \notin E$  must come from  $E^* \cup \{u_i : 1 \leq i \leq c\}$ , each of which maps to a distinct ball. Thus the second claim holds. By R26.Add.8x, since distinct words of the same length must be  $<_D$ -related, the first claim holds.

**Proposition R26.Add.10** Let  $k$  be in  $\mathbf{N}$  with  $k \geq 2$ . Then there is an auxiliary function  $\phi$  for  $\mathbf{R}_k$  such that  $\phi(v) \neq \emptyset$  for every  $v \in M_2$  and, if  $w$  is an infinite word,  $\text{diam}(\phi(w_n)) \rightarrow 0$  as  $n \rightarrow \infty$ .

Proof: First, a sequence of maps will be defined inductively. By R26.Add.1i  $\mathbf{R}_k$  can be written as  $\cup_{i=1}^k B_1(f_k(i))$ . Let  $j$  be the positive integer with  $2^j \leq k < 2^{j+1}$ . Apply R26.Add.8 with  $x = 1$  and  $l = 0$  to obtain  $\phi_1$  from  $D_1$  to the collection of unions from  $\{B_1(f_k(1)), \dots, B_1(f_k(k))\}$ , where  $\phi_1$  has the lemma's properties, which easily imply the 10 properties listed below. Now assume  $\phi_1, \dots, \phi_n$  with domains  $D_1 \subseteq D_2 \subseteq \dots \subseteq D_n$  have been constructed with the following properties:

- i)  $D_n \subseteq M_2$  is finite and  $\{w \in M_2 : \text{length}(w) \leq n\} \subseteq D_n$ .
- ii) For  $2 \leq i \leq n$ ,  $\phi_{i-1} = \phi_i|_{D_{i-1}}$ .
- iii)  $\phi_n(\eta) = \mathbf{R}_k$ .
- iv) If  $wv \in D_n$ , then  $w \in D_n$ .
- v) For  $w \in D_n$ ,  $w0 \in D_n$  if and only if  $w1 \in D_n$ .
- vi) For  $w \in D_n$  with  $w0 \in D_n$ ,  $\phi_n(w) = \phi_n(w0) \cup \phi_n(w1)$ .
- vii) If  $w \in D_n$  and  $w0 \notin D_n$ ,  $\text{diam}(\phi_n(w)) \leq 2/k^{n-1}$ .
- viii) For every  $w \in D_n$ ,  $\phi_n(w)$  is a non-empty closed set.
- ix) If  $w \in D_n$  and  $w0 \notin D_n$ ,  $\phi_n(w) = B_{1/k^{n-1}}(f_k(x(w)))$  for some  $x(w) \in \mathbf{N}$ .
- x) If  $w \in D_n$  and  $w = ab$ , then  $\phi_n(w) \subseteq \phi_n(a)$ .

Call  $w \in M_2$  a terminal word (of  $D_n$ ) if  $w \in D_n$  and  $w0 \notin D_n$ . There are finitely many terminal words. Note that no proper initial subword of a terminal word is a terminal word by properties iv) and v). For each terminal word  $w$ , by ix) and R26.Add.1ii,  $\phi_n(w) = \cup_{s=0}^{k-1} B_{1/k^n}(f_k(x(w) + sk^n))$  for some  $x(w) \in \mathbf{N}$ . By R26.Add.8 there is  $\psi_w$  from a finite domain  $A_w \subseteq M_2$  to the collection of unions from the family  $\{B_{1/k^n}(x(w) + sk^n)\}_{s=0}^{k-1}$ , where  $\psi_w$  has the properties listed in R26.Add.8. Let  $C_{n+1}$  denote the (finite) union over all terminal words of the finite sets  $\{wv : v \in A_w, v \neq \eta\}$ . Note that if  $wv = us$  where  $w, u$  are terminal words, then  $w = u$ , since neither can be a proper initial subword of the other, and so also  $v = s$ . Thus each element of  $C_{n+1}$  has a unique representation in the specified form. Finally note that the requirement  $v \neq \eta$  implies  $D_n \cap C_{n+1} = \emptyset$ . Let  $D_{n+1} = D_n \cup C_{n+1}$ . Define  $\phi_{n+1}$  on  $D_{n+1}$  by  $\phi_{n+1}(u) = \phi_n(u)$  if  $u \in D_n$  and, if  $u \in C_{n+1}$  with  $u = wv$ ,  $\phi(u) = \psi_w(v)$ . Then  $\phi_{n+1}$  is a function; clearly  $D_n \subseteq D_{n+1}$  and  $\phi_n = \phi_{n+1}|_{D_n}$  so that ii) holds for  $2 \leq i \leq n+1$ . Likewise, iii) and viii) hold for  $\phi_{n+1}$ . For i), finiteness holds since each  $A_w$  is finite. Let  $w$  be a word of length  $n+1$ . Then  $w = ux$ , where  $\text{length}(u) = n$  and  $x \in \{0, 1\}$ . Then  $u \in D_n$ . If  $u0 \in D_n$ ,  $w \in D_n \subseteq D_{n+1}$ . Otherwise  $u$  is a terminal word of  $D_n$  and, since  $\{0, 1\} \subseteq A_u$ ,  $w = ux \in C_{n+1} \subseteq D_{n+1}$ . For iv), let  $wv \in D_{n+1}$ . Since iv) holds for  $D_n$ , assume  $wv \in C_{n+1}$ . Then  $wv = us$ , where  $u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . If  $w$  is a subword of  $u$ ,  $u = wr$  and so  $w \in D_n$  by the induction hypothesis. Otherwise  $u$  must be a proper subword of  $w$ , i.e.,  $w = ut$  where  $t \neq \eta$ . Then  $wv = utv$  and so  $tv = s \in A_u$ . By R26.Add.2ii  $t \in A_u$

and so  $w = ut \in C_{n+1} \subseteq D_{n+1}$ . For v) let  $w \in D_{n+1}$  with  $w0 \in D_{n+1}$ . If  $w0 \in D_n$ , by iv) and v) of the induction hypothesis,  $w1 \in D_n \subseteq D_{n+1}$ . If  $w0 \in C_{n+1}$ ,  $w0 = us$ , where  $u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . Now  $s = r0$  and by R26.Add.8vi  $r1 \in A_u$ . Thus  $w1 = ur1$  is in  $C_{n+1} \subseteq D_{n+1}$ . The converse is similar. For vi) let  $w0$  (and so  $w$ ) be in  $D_{n+1}$ . If  $w0 \in D_n$ , since  $\phi_n = \phi_{n+1}|_{D_n}$ , apply vi) of the induction hypothesis for  $D_n$ . If  $w0 \in C_{n+1}$ ,  $w0 = us0$ , where  $u$  is a terminal word of  $D_n$  and  $s0 \in A_u$ . By R26.Add.8ii  $s \in A_u$  and so  $w = us \in C_{n+1}$ . Applying the definition and R26.Add.8vii,  $\phi_{n+1}(w) = \psi_u(s) = \psi_u(s0) \cup \psi_u(s1) = \phi_{n+1}(w0) \cup \phi_{n+1}(w1)$ . For ix) let  $w \in D_{n+1}$  with  $w0 \notin D_{n+1}$ . Note that  $w \notin D_n$  for, if so,  $w$  is a terminal word of  $D_n$  and  $0 \in A_w$  so that  $w0$  would be in  $C_{n+1} \subseteq D_{n+1}$ . Thus  $w = us$ , where  $u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . Since  $w0 = us0$  is not in  $C_{n+1}$ ,  $s0 \notin A_u$ . As above,  $\phi_n(u)$  is a union of balls of radius  $1/k^n$ . By R26.Add.8ix  $\psi_u(s) = \phi_{n+1}(w)$  is a ball of radius  $1/k^n$  as required for ix). It follows that  $\text{diam}(\phi_{n+1}(w)) \leq 2/k^n$  so that vii) also holds. For x) let  $w = ab$  be in  $D_{n+1}$ . By iv)  $a \in D_{n+1}$ . The conclusion is trivial if  $b = \eta$  and so assume  $b \neq \eta$  and proceed by induction on the length of  $b$ . For the initial case,  $b \in \{0, 1\}$ . By vi)  $\phi_{n+1}(a) = \phi_{n+1}(a0) \cup \phi_{n+1}(a1)$  and the conclusion holds. Assume it holds for any word of length  $r$  and let  $\text{length}(b) = r + 1$ . Write  $b$  as  $cx$  where  $x \in \{0, 1\}$  and  $\text{length}(c) = r$ . By the initial step  $\phi_{n+1}(w) \subseteq \phi_{n+1}(ac)$  and by the induction hypothesis  $\phi_{n+1}(ac) \subseteq \phi_{n+1}(a)$ . By induction there is an infinite sequence  $\{\phi_n\}_{n=1}^\infty$  with the properties listed above. Since the domains are nested and ii) holds,  $\phi = \bigcup_{n=1}^\infty \phi_n$  is a function and  $\phi|_{D_n} = \phi_n$  for every  $n$ . Since i) holds, the domain of  $\phi$  is  $M_2$ . For  $w \in M_2$ , pick  $n$  with  $w0 \in D_n$ . By vi)  $\phi(w) = \phi_n(w) = \phi_n(w0) \cup \phi_n(w1) = \phi(w0) \cup \phi(w1)$ . Thus  $\phi$  is an auxiliary function for  $\mathbf{R}_k$  and by viii)  $\phi(w) \neq \emptyset$  for every  $w$ . Lastly, let  $w$  be an infinite word and let  $\epsilon > 0$ . Pick  $N$  such that  $2/k^{N-1} < \epsilon$ . Since  $D_N$  is finite, the set  $\{n : w_n \notin D_N\}$  is non-empty and so has a smallest element  $M$ . Then  $M \geq N + 1$  by i) and  $w_{M-1} \in D_N$ . By v) and the choice of  $M$ ,  $w_{M-1}0 \notin D_N$ . By vii)  $\text{diam}(\phi(w_{M-1})) = \text{diam}(\phi_N(w_{M-1})) \leq 2/k^{N-1} < \epsilon$ . For  $n \geq M$ , by R26.11  $\phi(w_n) \subseteq \phi(w_{M-1})$  and so  $\text{diam}(\phi(w_n)) < \epsilon$ . Thus the limit claim holds.

**Lemma R26.Add.11** Let  $k$  be in  $\mathbf{N}$  with  $k \geq 2$ . Let  $\{\phi_n\}_{n=1}^\infty$  be the sequence of maps constructed in the previous proof. For every  $n$ , if  $u, v$  are equal length elements of  $D_n$  with  $u <_D v$ , then  $\phi_n(u) <_k \phi_n(v)$ .

*Proof:* Continue with the notation and terminology from the previous proof and proceed by induction on  $n$ . For  $n = 1$  the claim holds by R26.Add.8x and so assume it holds for  $n$  and let  $u, v$  be equal-length elements of  $D_{n+1}$  with  $u <_D v$ . There several cases to consider. If both  $u, v$  are in  $D_n$ ,  $\phi_{n+1}(u) = \phi_n(u) <_k \phi_n(v) = \phi_{n+1}(v)$ . Next suppose  $u \in D_n$  and  $v \in C_{n+1}$ . Let  $v = ws$ , where  $w$  is a terminal word of  $D_n$ . Since  $s \neq \eta$ ,  $w$  is shorter than  $v$  and  $u$ . Let  $u = u_1t$  where  $\text{length}(u_1) = \text{length}(w)$  so that  $t \neq \eta$ . By property iv) of  $\phi_n$ ,  $u_1$  and  $u_1x$  for some  $x \in \{0, 1\}$  are both in  $D_n$  and so by v)  $u_1$  is not a terminal word of  $D_n$ . Thus  $u_1 \neq w$  and so the dictionary ordering of  $u_1$  and  $w$  determines the ordering of  $u$  and  $v$ , i.e.,  $u_1 <_D w$ . By the induction hypothesis  $\phi_n(u_1) <_k \phi_n(w)$ . By property x) for  $\phi_{n+1}$ ,  $\phi_{n+1}(u) \subseteq \phi_{n+1}(u_1) = \phi_n(u_1)$  and  $\phi_{n+1}(v) \subseteq \phi_{n+1}(w) = \phi_n(w)$ . It follows easily that  $\phi_{n+1}(u) <_k \phi_{n+1}(v)$ . As a second case, assume  $u \in C_{n+1}$  and  $v \in D_n$ . Let  $u = yt$  where  $y$  is a terminal word of  $D_n$  and  $t \neq \eta$  so that  $y$  is shorter than both  $u$  and  $v$ . Write  $v = v_1s$  where  $\text{length}(v_1) = \text{length}(y)$  and so  $s \neq \eta$ . As in the first case  $v_1$  is in  $D_n$

but is not a terminal word so that  $v_1 \neq y$ , the ordering of  $v_1$  and  $y$  determines the ordering of  $v$  and  $u$ , and  $y <_D v_1$ . Using the induction hypothesis and x) again,  $\phi_n(y) <_k \phi_n(v_1)$ ,  $\phi_{n+1}(u) \subseteq \phi_{n+1}(y) = \phi_n(y)$  and  $\phi_{n+1}(v) \subseteq \phi_{n+1}(v_1) = \phi_n(v_1)$ . It follows easily that  $\phi_{n+1}(u) <_k \phi_{n+1}(v)$ . The last case, both  $u$  and  $v$  are in  $C_{n+1}$ , has several subcases. Write  $u = yt$  and  $v = ws$ , where  $y, w$  are both terminal elements of  $D_n$  and both  $s, t$  are non-null. If  $y = w$ ,  $t$  and  $s$  are equal-length length elements of the domain of  $\psi_y = \psi_w$ . The dictionary ordering of  $u$  and  $v$  is determined by the ordering of  $t$  and  $s$ , i.e.,  $t <_D s$ . By R26.Add.8x and the definition of  $\phi_{n+1}$ ,  $\phi_{n+1}(u) = \psi_y(t) <_k \psi_y(s) = \psi_w(s) = \phi_{n+1}(v)$ . Next suppose  $y \neq w$  and  $\text{length}(y) = \text{length}(w)$ . Here the ordering of  $u$  and  $v$  is determined by the order of  $y$  and  $w$ , i.e.,  $y <_D w$ . By the induction hypothesis and the definition of  $\phi_{n+1}$ ,  $\phi_{n+1}(y) = \phi_n(y) <_k \phi_n(w) = \phi_{n+1}(w)$ . By property x) for  $\phi_{n+1}$ ,  $\phi_{n+1}(u) \subseteq \phi_{n+1}(y)$  and  $\phi_{n+1}(v) \subseteq \phi_{n+1}(w)$ . The required conclusion now follows easily. For the next case suppose  $y \neq w$  and  $\text{length}(y) < \text{length}(w)$ . Write  $w = w_1q$ , where  $\text{length}(w_1) = \text{length}(y)$  and  $q \neq \eta$ . Since  $w \in D_n$  and  $q \neq \eta$ ,  $w_1 \in D_n$  but is not a terminal word. Thus  $w_1 \neq y$  and so the ordering of  $u, v$  is determined by  $y$  and  $w_1$ , i.e.,  $y <_D w_1$ . By the induction hypothesis,  $\phi_{n+1}(y) = \phi_n(y) <_k \phi_n(w_1) = \phi_{n+1}(w_1)$ . By property x) for  $\phi_{n+1}$ ,  $\phi_{n+1}(u) \subseteq \phi_{n+1}(y)$  and  $\phi_{n+1}(v) \subseteq \phi_{n+1}(w_1)$ , and so  $\phi_{n+1}(u) <_k \phi_{n+1}(v)$ . Finally, suppose  $y \neq w$  and  $\text{length}(y) > \text{length}(w)$ . Write  $y = y_1r$ , where  $\text{length}(y_1) = \text{length}(w)$  and  $r \neq \eta$ . As in the previous case,  $y_1$  is in  $D_n$  but is not a terminal word, and the ordering of  $u, v$  is determined by the ordering of  $y_1, w$  so that  $y_1 <_D w$ . Again by the induction hypothesis,  $\phi_{n+1}(y_1) = \phi_n(y_1) <_k \phi_n(w) = \phi_{n+1}(w)$ . By property x) for  $\phi_{n+1}$ ,  $\phi_{n+1}(u) \subseteq \phi_{n+1}(y_1)$  and  $\phi_{n+1}(v) \subseteq \phi_{n+1}(w)$ , and so  $\phi_{n+1}(u) <_k \phi_{n+1}(v)$ .

**Corollary R26.Add.12** Let  $k$  be in  $\mathbf{N}$  with  $k \geq 2$ . Let  $\phi$  be the auxiliary function for  $\mathbf{R}_k$  constructed as in the proof of R26.Add.10. If  $u, v$  are equal-length elements of  $M_2$  with  $u <_D v$ , then  $\phi(u) <_k \phi(v)$ .

Proof: Continue with the notation in the proof of R26.Add.10. Let  $u, v$  be equal-length elements of  $M_2$  with  $u <_D v$  and pick  $n$  greater than their common length so that  $u, v \in D_n$ . By R26.Add.11  $\phi(u) = \phi_n(u) <_k \phi_n(v) = \phi(v)$ .

**Proposition R26.Add.13** Let  $k$  be in  $\mathbf{N}$  with  $k \geq 2$ . There is an order-preserving bijection from  $(\mathcal{M}, <_D)$  to  $(\mathbf{R}_k, <_k)$ .

Proof: Let  $\phi$  be the auxiliary function constructed as in the proof of R26.Add.10. For  $w \in \mathcal{M}$ , let  $g(w) = \bigcap_{n=1}^{\infty} \phi(w_n)$ . As in the proof of R26.13  $g$  is a function and every element of  $\mathbf{R}_k$  is the image of some infinite word, i.e.,  $g$  is onto. Let  $u, w$  be in  $\mathcal{M}$  with  $u <_D w$ . By definition the dictionary order is determined by the first position where they differ and so there is  $n$  with  $u_n <_D w_n$ . By R26.Add.12  $\phi(u_n) <_k \phi(w_n)$ . Because  $g(u) \in \phi(u_n)$  and  $g(w) \in \phi(w_n)$ ,  $g(u) <_k g(w)$ . Thus  $g$  is order-preserving. Lastly, if  $u \neq w$ , either  $u <_D w$  or  $w <_D u$  so that  $g(u) <_k g(w)$  or  $g(w) <_k g(u)$ . In either case, because the inequality is strict,  $g(u) \neq g(w)$  and so  $g$  is one-to-one.

The next lemma records the simple and undoubtedly known fact that an order-preserving bijection between linearly ordered spaces must be an order isomorphism.

**Lemma R26.Add.14** Let  $(X, <)$  and  $(Y, <)$  be linearly ordered spaces. Let the map  $h : X \rightarrow Y$  be one-to-one, onto, and order-preserving. Then  $h^{-1}$  is also order-preserving.

Proof: Let  $y_1, y_2 \in Y$  with  $y_1 < y_2$  and let  $x_i = h^{-1}(y_i)$ . If  $x_2 \leq x_1$ , then, since  $h$  is order-preserving,  $h(x_2) \preceq h(x_1)$ , i.e.,  $y_2 \preceq y_1$ , a contradiction. Thus  $x_1 < x_2$  as required.

**Corollary R26.Add.15** Let  $k, l$  be in  $\mathbf{N}$  with  $k, l \geq 2$ . There is an order-preserving bijection from  $(\mathbf{R}_k, <_k)$  to  $(\mathbf{R}_l, <_l)$ .

Proof: Let  $g_k$  and  $g_l$  be order-preserving bijections from  $(\mathcal{M}, <_D)$  to  $(\mathbf{R}_k, <_k)$ , respectively  $(\mathbf{R}_l, <_l)$ . Then  $g_l \circ g_k^{-1}$  is the required map.

Next two more certainly known facts are recorded.

**Lemma R26.Add.16** Let  $(X, <)$  and  $(Y, \prec)$  be linearly ordered spaces. Let the map  $h : X \rightarrow Y$  be one-to-one, onto, and order-preserving. Let  $a, b$  be a consecutive pair in  $X$  with  $b$  larger. Then  $h(a), h(b)$  are consecutive in  $Y$  with  $h(b)$  larger.

Proof: Since  $a < b$ ,  $h(a) \prec h(b)$ . Suppose the images are not consecutive, i.e., there is  $y \in Y$  with  $h(a) \prec y \prec h(b)$ . Then  $a < h^{-1}(y) < b$ , which contradicts the assumption that  $a, b$  are consecutive.

**Lemma R26.Add.17** Let  $(X, <)$  and  $(Y, \prec)$  be linearly ordered spaces. Let the map  $h : X \rightarrow Y$  be an order-preserving bijection. Then  $h : (X, \tau(<)) \rightarrow (Y, \tau(\prec))$  is a homeomorphism.

Proof: Let  $y \in Y$ . It is easy to check that  $h^{-1}[(-\infty, y)] = (-\infty, h^{-1}(y))$  and  $h^{-1}[(y, \infty)] = (h^{-1}(y), \infty)$ . Thus the inverse image of every subbasic open set in  $Y$  is open in  $X$  and so  $h$  is continuous. Similarly,  $h^{-1}$  is continuous and the result follows.

Note that R26.23 could be derived from R26.Add.15, R26.Add.17, and R19.1.7.

**Lemma R26.Add.18** Let  $k, l$  be in  $\mathbf{N}$  with  $k, l \geq 2$  and let  $h$  be an order-preserving bijection from  $(\mathbf{R}_k, <_k)$  to  $(\mathbf{R}_l, <_l)$ . Then  $h[f_k[\mathbf{N}]] = f_l[\mathbf{N}]$ .

Proof: Since an order isomorphism maps the smallest element to the smallest,  $h(f_k(1))$  is  $f_l(1)$ . Now let  $j \in \mathbf{N}$  with  $j \geq 2$ . By R19.1.15  $f_k(j)$  is the larger of a consecutive pair. By R26.Add.15  $h(f_k(j))$  is the larger of a consecutive pair in  $\mathbf{R}_l$ . By R19.1.19 there is  $m$  in  $\mathbf{N}$  with  $m \geq 2$  such that  $h(f_k(j)) = f_l(m)$ . Thus  $h[f_k[\mathbf{N}]] \subseteq f_l[\mathbf{N}]$ . The same argument applied to  $h^{-1}$  yields the reverse containment.

**Corollary R26.Add.19** Let  $k, l$  be in  $\mathbf{N}$  with  $k, l \geq 2$ . Then the topological spaces  $\mathbf{N}_k$  and  $\mathbf{N}_l$  are homeomorphic.

Proof: This is immediate from R26.Add.15, R26.Add.17, R26.Add.18, and R26.28.

## Additional Reference

9. This website, R19: Ordering the Remnant Rings

## Added 2025

This added subsection presents a different construction of homeomorphisms, including an order-isomorphism, between  $\mathbf{R}_2$  and the Cantor set  $C$ .

Notation for  $C$  will as described before R26.30 above. In addition, for  $x \in C$ ,  $I_n(x)$  will denote the unique subinterval containing  $x$  in the union of pairwise disjoint closed subintervals making up  $C_n$ .  $I_n(x)$  may also be written in interval notation:  $I_n(x) = [a_n(x), b_n(x)]$ .

**Lemma R26.Add.20** Let  $x \in C$ . Then

- i) For  $n \in \mathbf{N}$ ,  $I_{n+1}(x) \subseteq I_n(x)$ .
- ii)  $\{x\} = \bigcap_{n+1}^{\infty} I_n(x)$ .
- iii)  $\lim_{n \rightarrow \infty} a_n(x) = \lim_{n \rightarrow \infty} b_n(x) = x$ .

Proof: For any  $n \in \mathbf{N}$ ,  $I_{n+1}(x)$  must be either the left third or the right third of  $I_n(x)$ . In either case i) follows. For ii) by definition  $x$  is in the intersection. For  $y \in C$  with  $y \neq x$ , pick  $n$  such that  $1/3^n < |x - y|$ . By R26.30  $|a_n(x) - b_n(x)| = 1/3^n$  and so, since  $x \in I_n(x)$ ,  $y$  is not in  $I_n(x)$ . Thus ii) holds. Because  $a_n(x) \leq x \leq b_n(x)$ , part iii) follows easily from R26.30.

As noted in the previous proof, for  $x \in C$ ,  $I_{n+1}(x)$  must be either the left third or the right third of  $I_n(x)$ . For convenience, given  $J = [a, b]$  with  $a < b$ ,  $l(J)$ , respectively  $r(J)$ , will denote the closed left, respectively right, third of  $J$ . Clearly  $l(J)$  and  $r(J)$  are both non-empty. Note that  $I_{n+1}(x)$  is either  $l(I_n(x))$  or  $r(I_n(x))$ .

**Definition R26.Add.21** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ . Let  $x \in C$ . The  $\sigma$ -sequence associated with  $x$  is defined inductively as follows:

$$x_1 = \begin{cases} 1 + \sigma(1) \cdot 2^0 & \text{if } x \in l([0, 1]); \\ 1 + (1 - \sigma(1)) \cdot 2^0 & \text{if } x \in r([0, 1]). \end{cases}$$

If  $x_n$  has been defined,

$$x_{n+1} = \begin{cases} x_n + \sigma(n+1) \cdot 2^n & \text{if } x \in l(I_n(x)); \\ x_n + (1 - \sigma(n+1)) \cdot 2^n & \text{if } x \in r(I_n(x)). \end{cases}$$

**Lemma R26.Add.22** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ , let  $x \in C$ , and let  $\{x_n\}_{n=1}^{\infty}$  be the  $\sigma$ -sequence associated with  $x$ . Then

- i) For  $n \in \mathbf{N}$ ,  $x_n \in \{1, \dots, 2^n\}$ .
- ii) For  $n \in \mathbf{N}$ , there is  $t \in \{0, 1\}$  such that  $x_{n+1} = x_n + t \cdot 2^n$ .
- iii) For  $n, j \in \mathbf{N}$  with  $j \leq n$ ,  $x_n \equiv x_j \pmod{2^j}$ .

Proof: For  $n \in \mathbf{N}$ , because  $\sigma(n) \in \{0, 1\}$ ,  $\{\sigma(n), 1 - \sigma(n)\} = \{0, 1\}$ . Thus part ii) holds by definition and  $x_1 \in \{1, 2\}$ . Now assume  $x_n \in \{1, \dots, 2^n\}$ . Then  $1 \leq x_n \leq x_{n+1} \leq 2^n + 2^n = 2^{n+1}$ . By induction i) holds. Part iii) clearly holds for  $n = 1$ . Assume it holds for  $n$ . Then  $x_{n+1} \equiv x_n \pmod{2^n}$  by ii) and so  $x_{n+1} \equiv x_n \pmod{2^j}$  for all  $j \leq n$ . By the induction hypothesis and the properties of equivalence relations, part iii) holds for  $n + 1$ .

**Lemma R26.Add.23** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ , let  $x, y \in C$ , and let  $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$  be the  $\sigma$ -sequences associated with  $x, y$  respectively. Then

- i) If  $x_m = y_m$  for some  $m \in \mathbf{N}$ , then  $x_n = y_n$  for all  $n \leq m$ .
- ii) Let  $m \in \mathbf{N}$ . Then  $x_m = y_m$  if and only if  $I_m(x) = I_m(y)$ .
- iii)  $x = y$  if and only if  $x_n = y_n$  for every  $n \in \mathbf{N}$ .

Proof: Parts i) and ii) are done by induction on  $m$ . Part i) clearly holds for  $m = 1$ . Assume it's true for  $m$  and  $x_{m+1} = y_{m+1}$ . By R26.Add.22ii there are  $r, s \in \{0, 1\}$  such that  $x_{m+1} = x_m + r \cdot 2^m$  and  $y_{m+1} = y_m + s \cdot 2^m$ . Since  $x_{m+1} = y_{m+1}$ ,  $|x_m - y_m| = |s - r| \cdot 2^m$ , i.e.,  $|x_m - y_m|$  is  $2^m$  if  $r \neq s$  and 0 if  $r = s$ . By R26.Add.22i  $|x_m - y_m| \leq 2^m - 1 < 2^m$  and so  $x_m = y_m$ . By the induction hypothesis  $x_n = y_n$  for all  $n \leq m$  and so part i) holds for  $m + 1$ . For part ii): By the definition of  $\sigma$ -sequence,  $x_1 = y_1$  if and only if  $x$  and  $y$  are both in  $l([0, 1])$ , i.e., both in  $[0, 1/3]$ , or both in  $r([0, 1])$ , i.e., both in  $[2/3, 1]$ . Either case holds if and only if  $I_1(x) = I_1(y)$ . Now assume ii) holds for  $m$  and  $x_{m+1} = y_{m+1}$ . Part i) implies  $x_m = y_m$ , and so  $I_m(x) = I_m(y)$  by the induction hypothesis. These equalities and the definition of  $\sigma$ -sequence imply  $x, y$  are both to the left in  $I_m(x) = I_m(y)$  or both to the right.

In either case  $I_{m+1}(x) = I_{m+1}(y)$ . For the converse assume  $I_{m+1}(x) = I_{m+1}(y)$ . Since  $y \in I_{m+1}(x) \subseteq I_m(x)$ , by definition  $I_m(y) = I_m(x)$  and so, by the induction hypothesis,  $x_m = y_m$ . By the definition of the  $\sigma$ -sequence, since  $I_m(x) = I_m(y)$ ,  $I_{m+1}(x) = I_{m+1}(y)$ , and  $x_m = y_m$ ,  $x_{m+1} = y_{m+1}$ . Thus ii) holds. For part iii), if  $x = y$ , by definition  $x_n = y_n$  for all  $n$ . Now assume  $x \neq y$  and pick  $m$  such that  $1/3^m < |x - y|$ . By R26.30 the length of  $I_m(x)$  is  $1/3^m$ . Since  $x \in I_m(x)$ ,  $y \notin I_m(x)$ , i.e.,  $I_m(x) \neq I_m(y)$ . By part ii)  $x_m \neq y_m$ .

**Lemma R26.Add.24** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$  and let  $\{z_n\}_{n=1}^{\infty}$  be a sequence such that  $z_n \in \{1, \dots, 2^n\}$  for every  $n \in \mathbf{N}$  and  $n \in \mathbf{N}$  implies there is  $t(n) \in \{0, 1\}$  such that  $z_{n+1} = z_n + t(n) \cdot 2^n$ . Then there is  $z \in C$  such that  $\{z_n\}_{z=1}^{\infty}$  is the  $\sigma$ -sequence associated with  $z$ .

Proof: First, a sequence of closed subintervals of  $[0, 1]$  will be defined inductively.

$$J_1 = \begin{cases} l([0, 1]) & \text{if } (\sigma(1) = 0 \text{ and } z_1 = 1) \text{ or } (\sigma(1) = 1 \text{ and } z_1 = 2); \\ r([0, 1]) & \text{if } (\sigma(1) = 0 \text{ and } z_1 = 2) \text{ or } (\sigma(1) = 1 \text{ and } z_1 = 1). \end{cases}$$

Now assume  $J_n$  has been defined.

$$J_{n+1} = \begin{cases} l(J_n) & \text{if } (\sigma(n+1) = 0 \text{ and } z_{n+1} = z_n) \text{ or } (\sigma(n+1) = 1 \text{ and } z_{n+1} = z_n + 2^n); \\ r(J_n) & \text{if } (\sigma(n+1) = 0 \text{ and } z_{n+1} = z_n + 2^n) \text{ or } (\sigma(n+1) = 1 \text{ and } z_{n+1} = z_n). \end{cases}$$

Note that  $J_{n+1} \subseteq J_n$  for all  $n$ , each  $J_n$  is one of the closed, non-empty subintervals in the union making up  $C_n$ , and by R26.30 the length of  $J_n$  is  $1/3^n$ . The nesting implies that  $\{J_n : n \in \mathbf{N}\}$  has the finite intersection property and so by compactness  $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$ . Since the lengths tend to zero, the intersection must be a singleton. Let  $\bigcap_{n=1}^{\infty} J_n = \{z\}$ . Since  $J_n \subseteq C_n$  for every  $n$ ,  $z \in C$ . By definition  $I_n(z) = J_n$  for every  $n$ . Let  $\{x_n\}_{n=1}^{\infty}$  be the  $\sigma$ -sequence associated with  $z$ . The conclusion follows if  $x_n = z_n$  for every  $n$ , which will be shown by induction. The definition above and R26.Add.21 are used repeatedly in the multiple cases. First assume  $n = 1$ . For  $z_1 = 1$ : If  $\sigma(1) = 0$ ,  $J_1 = l([0, 1])$  so that  $x_1 = 1 + \sigma(1) \cdot 2^0 = 1 = z_1$ . If  $\sigma(1) = 1$ , then  $J_1 = r([0, 1])$  so that  $x_1 = 1 + (1 - \sigma(1)) \cdot 2^0 = 1 = z_1$ . For  $z_1 = 2$ : If  $\sigma(1) = 0$ ,  $J_1 = r([0, 1])$  so that  $x_1 = 1 + (1 - \sigma(1)) \cdot 2^0 = 2 = z_1$ . If  $\sigma(1) = 1$ ,  $J_1 = l([0, 1])$  so that  $x_1 = 1 + \sigma(1) \cdot 2^0 = 2 = z_1$ . Now assume  $z_n = x_n$  for some  $n$ . For  $z_{n+1} = z_n$ : If  $\sigma(n+1) = 0$ ,  $J_{n+1} = l(J_n) = l(I_n(z))$  so that  $x_{n+1} = x_n + \sigma(n+1) \cdot 2^n = x_n = z_n = z_{n+1}$ . If  $\sigma(n+1) = 1$ ,  $J_{n+1} = r(J_n) = r(I_n(z))$  so that  $x_{n+1} = x_n + (1 - \sigma(n+1)) \cdot 2^n = x_n = z_n = z_{n+1}$ . For  $z_{n+1} = z_n + 2^n$ : If  $\sigma(n+1) = 0$ ,  $J_{n+1} = r(J_n) = r(I_n(z))$  so that  $x_{n+1} = x_n + (1 - \sigma(n+1)) \cdot 2^n = x_n + 2^n = z_n + 2^n = z_{n+1}$ . If  $\sigma(n+1) = 1$ ,  $J_{n+1} = l(J_n) = l(I_n(z))$  so that  $x_{n+1} = x_n + \sigma(n+1) \cdot 2^n = x_n + 2^n = z_n + 2^n = z_{n+1}$ .

**Lemma R26.Add.25** Let  $n \in \mathbf{N}$ . Let  $I, J$  be intervals in the union making up  $C_n$  with  $I \neq J$ . Suppose  $x \in I$  and  $y \in J$ . Then  $|x - y| \geq 1/3^n$ .

Proof: By induction: For  $n = 1$ , the two intervals must be  $[0, 1/3]$  and  $[2/3, 1]$ , which are closest at the inner endpoints so that the inequality holds. Now assume the assertion holds for some  $n \in \mathbf{N}$  and that  $I, J$  be intervals in the union making up  $C_{n+1}$  with  $I \neq J$ . Let  $x \in I$  and  $y \in J$ . By the definition of  $C_{n+1}$  there exist  $I^*, J^*$  be intervals in the union making up  $C_n$  with  $I \subseteq I^*$  and  $J \subseteq J^*$ . If  $I^* \neq J^*$ , by the induction hypothesis  $|x - y| \geq 1/3^n > 1/3^{n+1}$ . If  $I^* = J^* = [a, b]$ , where  $|a - b| = 1/3^n$  by R26.30, the two

intervals must be  $[a, a + (b - a)/3]$  and  $[b - (b - a)/3, b]$ , which are closet at the two inner endpoints. Thus  $|x - y| \geq |a - b|/3 = 1/3^{n+1}$ .

**Lemma R26.Add.26** Let  $t \in C$  and  $\{t_j\}$  be a sequence in  $C$ . Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ , let  $\{x_n\}_{n=1}^{\infty}$  be the  $\sigma$ -sequence associated with  $t$ , and  $\{x_n^j\}_{n=1}^{\infty}$  the  $\sigma$ -sequence associated with  $t_j$ . Then  $t_j \rightarrow t$  if and only if for every  $n \in \mathbf{N}$  there is  $M(n)$  such that  $j \geq M(n)$  implies  $x_n^j = x_n$ .

Proof: Assume  $t_j \rightarrow t$  and let  $n \in \mathbf{N}$ . There is  $M(n)$  such that  $j \geq M(n)$  implies  $|t_j - t| < 1/3^n$ . Let  $j \geq M(n)$ . By the previous lemma  $I_n(t_j) = I_n(t)$  and by R26.Add.23ii  $x_n^j = x_n$ . Conversely, assume the condition and let  $\epsilon > 0$ . Pick  $n \in \mathbf{N}$  such that  $1/3^n < \epsilon$ . By hypothesis for this part, there is  $M(n)$  such that  $j \geq M(n)$  implies  $x_n = x_n^j$ . Let  $j \geq M(n)$ . By R26.Add.23ii  $I_n(t_j) = I_n(t)$ . Both  $t_j$  and  $t$  are in  $I_n(t)$ , which has length  $1/3^n$  by R26.30. Thus  $|t_j - t| < \epsilon$ .

The next lemma is a redundant summary, but it makes clear that the subsequent definition does define a function. It uses the unique associated sequence of an element of  $\mathbf{R}_2$  as defined in R10.2.3.

**Lemma R26.Add.27** Let  $\mathcal{F} \in \mathbf{R}_2$  and let  $\{x_n\}_{n=1}^{\infty}$  be the associated sequence of  $\mathcal{F}$ . Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ . There is a unique  $x$  in  $C$  such that  $\{x_n\}_{n=1}^{\infty}$  is the  $\sigma$ -sequence of  $x$ .

Proof: For every  $n \in \mathbf{N}$ , by R10.2.3  $x_n \in \{1, \dots, 2^n\}$  and by R10.2.5i  $x_{n+1} = x_n + j(n) \cdot 2^n$  for some  $j(n) \in \{0, 1\}$ . By R26.Add.24 there is  $x \in C$  such that  $\{x_n\}_{n=1}^{\infty}$  is the  $\sigma$ -sequence of  $x$ . By R26.Add.23iii  $x$  is unique.

**Definition R26.Add.28** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ . The map  $\psi_\sigma : \mathbf{R}_2 \rightarrow C$  is defined by  $\psi_\sigma(\mathcal{F}) = x$ , where the  $\sigma$ -sequence of  $x$  is the associated sequence of  $\mathcal{F}$ .

**Proposition R26.Add.29** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ . The map  $\psi_\sigma : \mathbf{R}_2 \rightarrow C$  is a homeomorphism.

Proof: Let  $x \in C$  with  $\{x_n\}_{n=1}^{\infty}$  the  $\sigma$ -sequence of  $x$ . By R26.Add.22 and R10.2.6 there is  $\mathcal{F} \in \mathbf{R}_2$  such that the associated sequence of  $\mathcal{F}$  is  $\{x_n\}_{n=1}^{\infty}$ . By definition  $\psi_\sigma(\mathcal{F}) = x$  and so  $\psi_\sigma$  is onto. Now suppose  $\mathcal{F}, \mathcal{G} \in \mathbf{R}_2$  with  $\psi_\sigma(\mathcal{F}) = \psi_\sigma(\mathcal{G})$ . Let  $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$  be the associated sequences of  $\mathcal{F}, \mathcal{G}$  respectively. These sequences are the  $\sigma$ -sequences of the common value in  $C$ . By definition each element in  $C$  has a unique  $\sigma$ -sequence, i.e.,  $x_n = y_n$  for every  $n \in \mathbf{N}$ . By R10.2.4  $\mathcal{F} = \mathcal{G}$ . Thus  $\psi_\sigma$  is one-to-one. For continuity first note that  $\mathbf{R}_2$  is metrizable by R12.6.5 and so it is sufficient to verify that  $\psi_\sigma$  preserves sequence convergence. Let  $\{\mathcal{F}_j\}$  be a sequence in  $\mathbf{R}_2$  and assume  $\{x_n^j\}_{n=1}^{\infty}$  is the associated sequence of  $\mathcal{F}_j$ . Let  $\mathcal{F}$  in  $\mathbf{R}_2$  have associated sequence  $\{x_n\}_{n=1}^{\infty}$  and assume  $\mathcal{F}_i \rightarrow \mathcal{F}$ . Let  $n \in \mathbf{N}$ . By R17.2.16 there is  $M(n)$  such that  $j \geq M(n)$  implies  $x_n^j = x_n$ . By R26.Add.26 this condition for the  $\sigma$ -sequences implies  $\psi_\sigma(\mathcal{F}_i) \rightarrow \psi_\sigma(\mathcal{F})$ . Thus  $\psi_\sigma$  is continuous.  $C$  is  $T_2$  and by R12.6.5  $\mathbf{R}_2$  is compact. Thus the continuous bijection  $\psi_\sigma$  is a homeomorphism.

**Proposition R26.Add.30** Let  $\sigma$  and  $\delta$  be maps from  $\mathbf{N}$  to  $\{0, 1\}$  with  $\sigma \neq \delta$ . For every  $x \in C$ ,  $\psi_\sigma^{-1}(x) \neq \psi_\delta^{-1}(x)$ .

Proof: Let  $x \in C$ , let  $\{x_n\}_{n=1}^{\infty}$  be the  $\sigma$ -sequence of  $x$ , and let  $\{y_n\}_{n=1}^{\infty}$  be the  $\delta$ -sequence of  $x$ . Let  $m$  be the smallest element of the non-empty set  $\{n \in \mathbf{N} : \sigma(n) \neq \delta(n)\}$ . If  $m = 1$ , there are two cases. If  $x \in l([0, 1])$ , by definition  $x_1 - y_1 = (\sigma(1) - \delta(1)) \cdot 2^0 \neq 0$ . If  $x \in r([0, 1])$ , by definition  $x_1 - y_1 = (\delta(1) - \sigma(1)) \cdot 2^0 \neq 0$ . Thus  $x_1 \neq y_1$  and so  $\psi_\sigma^{-1}(x) \neq \psi_\delta^{-1}(x)$  by the uniqueness of associated sequences in  $\mathbf{R}_2$ . If  $m > 1$ ,  $\sigma(n) = \delta(n)$  for all  $n < m$  and so, by definition,  $x_n = y_n$  for all  $n < m$ . In particular  $x_{m-1} = y_{m-1}$ . Again

there are two cases. If  $x \in l(I_{m-1}(x))$ ,  $x_m - y_m = (\sigma(m) - \delta(m)) \cdot 2^{m-1} \neq 0$  by definition R26.Add.21. If  $x \in r(I_{m-1}(x))$ , by same definition  $x_m - y_m = (\delta(m) - \sigma(m)) \cdot 2^{m-1} \neq 0$ . In both cases  $x_m \neq y_m$  and so as before  $\psi_\sigma^{-1}(x) \neq \psi_\delta^{-1}(x)$ .

**Corollary R26.Add.31** Let  $\sigma$  and  $\delta$  be maps from  $\mathbf{N}$  to  $\{0, 1\}$  with  $\sigma \neq \delta$ . For every  $\mathcal{F} \in \mathbf{R}_2$ ,  $\psi_\sigma(\mathcal{F}) \neq \psi_\delta(\mathcal{F})$ .

Proof: Let  $\mathcal{F} \in \mathbf{R}_2$  and suppose  $\psi_\sigma(\mathcal{F}) = x = \psi_\delta(\mathcal{F})$ . Then  $\psi_\sigma^{-1}(x) = \mathcal{F} = \psi_\delta^{-1}(x)$ , which contradicts R26.Add.30.

**Corollary R26.Add.32** Let  $\sigma$  and  $\delta$  be maps from  $\mathbf{N}$  to  $\{0, 1\}$ . Then  $\psi_\sigma = \psi_\delta$  if and only if  $\sigma = \delta$ .

Proof: If  $\sigma = \delta$ , for each  $x \in C$  its  $\sigma$ -sequence and  $\delta$ -sequence are identical and so  $\psi_\sigma^{-1}(x) = \psi_\delta^{-1}(x)$ . Thus  $\psi_\sigma^{-1} = \psi_\delta^{-1}$ , i.e.,  $\psi_\sigma = \psi_\delta$ . By R26.Add.30, if  $\sigma \neq \delta$ ,  $\psi_\sigma^{-1} \neq \psi_\delta^{-1}$ , i.e.,  $\psi_\sigma \neq \psi_\delta$ .

**Corollary R26.Add.33** Let  $S$  be the set of homeomorphisms from  $\mathbf{R}_2$  onto  $C$ . Then  $|S| \geq 2^{\aleph_0}$ , i.e.,  $S$  is uncountable.

Proof: By R26.Add.29 and R26.Add.32  $\sigma \mapsto \psi_\sigma$  is an injection from  $\{0, 1\}^{\mathbf{N}}$  into  $S$ . Thus  $|S| \geq |\{0, 1\}^{\mathbf{N}}| = 2^{\aleph_0}$ .

The next lemma will be used in the subsequent example. The three sequences in the hypothesis identify distinct elements of  $\mathbf{R}_2$  by R10.2.6.

**Lemma R26.Add.34** Let  $\gamma, \delta : \mathbf{N} \rightarrow \{0, 1\}$  with  $\gamma(n) = 0$  for all  $n$ ,  $\delta(1) = 1$ , and  $\delta(n) = 0$  for all  $n \geq 2$ . Let  $\mathcal{F} \in \mathbf{R}_2$  be associated with  $\{x_n\}_{n=1}^\infty$  where  $x_n = 1$  for all  $n$ . Let  $\mathcal{G} \in \mathbf{R}_2$  be associated with  $\{y_n\}_{n=1}^\infty$  where  $y_n = 2$  for all  $n$ . Let  $\mathcal{H} \in \mathbf{R}_2$  be associated with  $\{z_n\}_{n=1}^\infty$  where  $z_1 = 1$  and  $z_n = 3$  for all  $n \geq 2$ . Then

- i)  $\psi_\gamma(\mathcal{G}) = 2/3$ .
- ii)  $\psi_\delta(\mathcal{F}) = 2/3$ .
- iii)  $\psi_\gamma(\mathcal{H}) = 2/9$ .
- iv)  $\psi_\delta(\mathcal{G}) \neq 2/9$ .

Proof: For i): The  $\gamma$ -sequence of  $\psi_\gamma(\mathcal{G})$  is  $\{y_n\}_{n=1}^\infty$ . It will be shown by induction that  $2/3$  is the left endpoint of  $I_n(\psi_\gamma(\mathcal{G}))$  for every  $n$ . By definition R26.Add.21, since  $y_1 = 2$  and  $\gamma(1) = 0$ ,  $\psi_\gamma(\mathcal{G}) \in r([0, 1]) = [2/3, 1] = I_1(\psi_\gamma(\mathcal{G}))$ . Now assume the claim is true for  $n$ . Since  $y_{n+1} = 2 = y_n$  and  $\gamma(n+1) = 0$ , by definition again  $\psi_\gamma(\mathcal{G})$  is in  $l(I_n(\psi_\gamma(\mathcal{G}))) = I_{n+1}(\psi_\gamma(\mathcal{G}))$ . The induction hypothesis now implies  $2/3$  is the left endpoint of  $I_{n+1}(\psi_\gamma(\mathcal{G}))$ . Thus  $\bigcap_{n=1}^\infty I_n(\psi_\gamma(\mathcal{G})) = \{2/3\}$ , i.e.,  $\psi_\gamma(\mathcal{G}) = 2/3$ . For ii): The  $\delta$ -sequence of  $\psi_\delta(\mathcal{F})$  is  $\{x_n\}_{n=1}^\infty$ . By a similar argument,  $2/3$  is the left endpoint of  $I_n(\psi_\delta(\mathcal{F}))$  for all  $n$ , which implies ii). For iii): The  $\gamma$ -sequence of  $\psi_\gamma(\mathcal{H})$  is  $\{z_n\}_{n=1}^\infty$ . Since  $z_1 = 1$  and  $\gamma(1) = 0$ ,  $\psi_\gamma(\mathcal{H}) \in l([0, 1]) = [0, 1/3] = I_1(\psi_\gamma(\mathcal{H}))$ . Since  $z_2 = 3$  and  $\gamma(2) = 0$ ,  $\psi_\gamma(\mathcal{H}) \in r(I_1(\psi_\gamma(\mathcal{H}))) = [2/9, 1, 3] = I_2(\psi_\gamma(\mathcal{H}))$ . Since  $z_{n+1} = z_n = 3$  for  $n \geq 2$  and  $\gamma(n) = 0$  for all  $n$ , continue by induction as above to see that  $2/9 \in I_n(\psi_\gamma(\mathcal{H}))$  for all  $n$ . Thus  $\psi_\gamma(\mathcal{H}) = 2/9$ . For iv): The  $\delta$ -sequence of  $\psi_\delta(\mathcal{G})$  is  $\{y_n\}_{n=1}^\infty$ . Since  $y_1 = 2$  and  $\delta(1) = 1$ ,  $\psi_\delta(\mathcal{G}) \in l([0, 1]) = [0, 1/3] = I_1(\psi_\delta(\mathcal{G}))$ . Since  $y_2 = 2$  and  $\delta(2) = 0$ ,  $\psi_\delta(\mathcal{G})$  is in  $l(I_1(\psi_\delta(\mathcal{G}))) = [0, 1/9]$ . Thus  $\psi_\delta(\mathcal{G}) \neq 2/9$ .

The following example shows that not every homeomorphism from  $\mathbf{R}_2$  to  $C$  is of the form  $\psi_\sigma$  for some  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ .

**Example R26.Add.35** Let  $\mathcal{F} \in \mathbf{R}_2$  be associated with  $\{x_n\}_{n=1}^\infty$  where  $x_n = 1$  for all  $n$ . Define  $h : \mathbf{R}_2 \rightarrow \mathbf{R}_2$  by  $h(\mathcal{E}) = \mathcal{E} + \mathcal{F}$ , where addition on  $\mathbf{R}_2$  is as described in [4]. By

R12.6.5  $h$  is a homeomorphism. Let  $\gamma : \mathbf{N} \rightarrow \{0, 1\}$  with  $\gamma(n) = 0$  for all  $n$ . Then  $\psi_\gamma \circ h$  is a homeomorphism. It will be shown  $\psi_\gamma \circ h \neq \psi_\sigma$  for every  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ . Let  $h(\mathcal{F}) = \mathcal{G}$  and let  $\mathcal{G}$  be associated with  $\{y_n\}_{n=1}^\infty$ . By R12.4.4  $y_n = 2$  for all  $n$  and so by R26.Add.34i  $\psi_\gamma \circ h(\mathcal{F}) = \psi_\gamma(\mathcal{G}) = 2/3$ . Let  $\delta : \mathbf{N} \rightarrow \{0, 1\}$  with  $\delta(1) = 1$  and  $\delta(n) = 0$  for all  $n \geq 2$ . For any  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$  with  $\sigma \neq \delta$ , since  $\psi_\delta(\mathcal{F}) = 2/3$ ,  $\psi_\sigma(\mathcal{F}) \neq 2/3$  by R26.Add.31 and so  $\psi_\gamma \circ h \neq \psi_\sigma$ . Now let  $\mathcal{H} = h(\mathcal{G}) = \mathcal{G} + \mathcal{F}$  and let  $\mathcal{H}$  be associated with  $\{z_n\}_{n=1}^\infty$ . By R12.4.4 again  $z_1 = 1$  and  $z_n = 3$  for all  $n \geq 2$ . By R26.Add.34iii  $\psi_\gamma \circ h(\mathcal{G}) = \psi_\gamma(\mathcal{H}) = 2/9$ . Since  $\psi_\delta(\mathcal{G}) \neq 2/9$  by R26.Add.34iv,  $\psi_\gamma \circ h \neq \psi_\delta$ . Thus  $\psi_\gamma \circ h \neq \psi_\sigma$  for all  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ .

**Proposition R26.Add.36** Let  $\gamma : \mathbf{N} \rightarrow \{0, 1\}$  by  $\gamma(n) = 0$  for all  $n$ . Then  $\psi_\gamma$  is order-preserving, i.e., if  $\mathcal{F}, \mathcal{G} \in \mathbf{R}_2$  with  $\mathcal{F} <_2 \mathcal{G}$ , then  $\psi_\gamma(\mathcal{F}) < \psi_\gamma(\mathcal{G})$ .

Proof: Let  $\mathcal{F}, \mathcal{G} \in \mathbf{R}_2$  with  $\mathcal{F} <_2 \mathcal{G}$ , and let the associated sequences of  $\mathcal{F}$  and  $\mathcal{G}$  be  $\{x_n\}$  and  $\{y_n\}$  respectively. Since  $\mathcal{F} \neq \mathcal{G}$ , the set  $\{n : x_n \neq y_n\}$  is non-empty and so has a smallest element  $m$ . By R19.Add.2  $x_m < y_m$ . By the definition of  $\psi_\gamma$ , the  $\gamma$ -sequences of  $\psi_\gamma(\mathcal{F})$  and  $\psi_\gamma(\mathcal{G})$  are  $\{x_n\}$  and  $\{y_n\}$  respectively. If  $m = 1$ ,  $x_1 = 1$  and  $y_1 = 2$ . By the definition of the  $\gamma$ -sequence,  $\psi_\gamma(\mathcal{F}) \in [0, 1/3]$  and  $\psi_\gamma(\mathcal{G}) \in [2/3, 1]$  so that  $\psi_\gamma(\mathcal{F}) < \psi_\gamma(\mathcal{G})$ . If  $m > 1$ , by the choice of  $m$ ,  $x_{m-1} = y_{m-1}$  and so by R26.Add.23ii  $I_{m-1}(\psi_\gamma(\mathcal{F})) = I_{m-1}(\psi_\gamma(\mathcal{G}))$ . Since  $x_m < y_m$ ,  $x_m = x_{m-1}$  and  $y_m = x_{m-1} + 2^{m-1}$ . By the definition of the  $\gamma$ -sequence,  $\psi_\gamma(\mathcal{F}) \in l(I_{m-1}(\psi_\gamma(\mathcal{F})))$  and  $\psi_\gamma(\mathcal{G}) \in r(I_{m-1}(\psi_\gamma(\mathcal{F})))$ . Thus  $\psi_\gamma(\mathcal{F}) < \psi_\gamma(\mathcal{G})$ .

Comment:  $C$  has been implicitly assumed to have the subspace topology from  $[0, 1]$ . The inherited order, written  $<_C$  for this comment, also defines a topology on  $C$ . It is easy to check that an order-preserving bijection of linearly ordered spaces must be a homeomorphism with the order topologies. Since the order topology generated by  $<_2$  is the topology of  $\mathbf{R}_2$  by R19.1.7, it follows from R26.Add.29 and R26.Add.36 that  $\tau(<_C)$  is the subspace topology on  $C$ . This would also follow because  $C$  is a closed subset of an order-complete space.

**Proposition R26.Add.37** Let  $\gamma : \mathbf{N} \rightarrow \{0, 1\}$  by  $\gamma(n) = 0$  for all  $n$ . Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$  with  $\sigma \neq \gamma$ . Then  $\psi_\sigma$  is not order-preserving.

Proof: Let  $m$  be the smallest of the non-empty set  $\{m : \sigma(n) \neq \gamma(n)\}$ . In what follows elements of  $\mathbf{R}_2$  are repeatedly identified by using R10.2.6. If  $m = 1$ , let  $\mathcal{F}$  in  $\mathbf{R}_2$  be associated with the sequence  $\{x_n\}$  where  $x_n = 1$  for all  $n$  and let  $\mathcal{G}$  in  $\mathbf{R}_2$  be associated with the sequence  $\{y_n\}$  where  $y_n = 2$  for all  $n$ . By R19.Add.2  $\mathcal{F} <_2 \mathcal{G}$ . Since  $\{x_n\}$  is the  $\sigma$ -sequence of  $\psi_\sigma(\mathcal{F})$  and  $\sigma(1) = 1$ , by the definition of  $\sigma$ -sequence  $\psi_\sigma(\mathcal{F}) \in r([0, 1])$ . Similarly,  $\psi_\sigma(\mathcal{G}) \in l([0, 1])$ . Thus  $\psi_\sigma(\mathcal{F}) > \psi_\sigma(\mathcal{G})$ . If  $m > 1$ , let  $\mathcal{F}$  in  $\mathbf{R}_2$  be associated with the sequence  $\{x_n\}$  where  $x_n = 1$  for all  $n$  and let  $\mathcal{G}$  in  $\mathbf{R}_2$  be associated with the sequence  $\{y_n\}$  where  $y_n = 1$  for  $n \leq m - 1$  and  $y_n = 1 + 2^{m-1}$  for  $n \geq m$ . Again by R19.Add.2  $\mathcal{F} <_2 \mathcal{G}$  and by definition of  $\psi_\sigma$ ,  $\{x_n\}$  is the  $\sigma$ -sequence of  $\psi_\sigma(\mathcal{F})$  and  $\{y_n\}$  is the  $\sigma$ -sequence of  $\psi_\sigma(\mathcal{G})$ . Since  $x_{m-1} = y_{m-1}$ , by R26.Add.23.ii  $I_{m-1}(\psi_\sigma(\mathcal{F})) = I_{m-1}(\psi_\sigma(\mathcal{G}))$ . Since  $\sigma(m) = 1$ , by definition of the  $\sigma$ -sequence,  $\psi_\sigma(\mathcal{F}) \in r(I_{m-1}(\psi_\sigma(\mathcal{F})))$  and  $\psi_\sigma(\mathcal{G})$  is in  $l(I_{m-1}(\psi_\sigma(\mathcal{F})))$ . Thus  $\psi_\sigma(\mathcal{F}) > \psi_\sigma(\mathcal{G})$  and the conclusion follows.

The corresponding results for order-reversing maps have similar proofs.

**Proposition R26.Add.38** Let  $\beta : \mathbf{N} \rightarrow \{0, 1\}$  by  $\beta(n) = 1$  for all  $n$ . Then  $\psi_\beta$  is order-reversing, i.e., if  $\mathcal{F}, \mathcal{G} \in \mathbf{R}_2$  with  $\mathcal{F} <_2 \mathcal{G}$ , then  $\psi_\beta(\mathcal{F}) > \psi_\beta(\mathcal{G})$ .

Proof: Let  $\mathcal{F}, \mathcal{G} \in \mathbf{R}_2$  with  $\mathcal{F} <_2 \mathcal{G}$ , and let the associated sequences of  $\mathcal{F}$  and  $\mathcal{G}$  be

$\{x_n\}$  and  $\{y_n\}$  respectively. Since  $\mathcal{F} \neq \mathcal{G}$ , the set  $\{n : x_n \neq y_n\}$  is non-empty and so has a smallest element  $m$ . By R19.Add.2  $x_m < y_m$ . By the definition of  $\psi_\beta$ , the  $\beta$ -sequences of  $\psi_\beta(\mathcal{F})$  and  $\psi_\beta(\mathcal{G})$  are  $\{x_n\}$  and  $\{y_n\}$  respectively. If  $m = 1$ ,  $x_1 = 1$  and  $y_1 = 2$ . By the definition of the  $\beta$ -sequence, since  $\beta(1) = 1$ ,  $\psi_\beta(\mathcal{F}) \in r([0, 1])$  and  $\psi_\beta(\mathcal{G}) \in l([0, 1])$ . Thus  $\psi_\beta(\mathcal{F}) > \psi_\beta(\mathcal{G})$ . If  $m > 1$ , by the choice of  $m$ ,  $x_{m-1} = y_{m-1}$  and so by R26.Add.23ii  $I_{m-1}(\psi_\beta(\mathcal{F})) = I_{m-1}(\psi_\beta(\mathcal{G}))$ . Since  $x_m < y_m$ ,  $x_m = x_{m-1}$  and  $y_m = x_{m-1} + 2^{m-1}$ . By the definition of the  $\beta$ -sequence,  $\psi_\beta(\mathcal{F}) \in r(I_{m-1}(\psi_\beta(\mathcal{F})))$  and  $\psi_\beta(\mathcal{G}) \in l(I_{m-1}(\psi_\beta(\mathcal{F})))$ . Thus  $\psi_\beta(\mathcal{F}) > \psi_\beta(\mathcal{G})$ .

**Proposition R26.Add.39** Let  $\beta : \mathbf{N} \rightarrow \{0, 1\}$  by  $\beta(n) = 1$  for all  $n$ . Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$  with  $\sigma \neq \beta$ . Then  $\psi_\sigma$  is not order-reversing.

Proof: Let  $m$  be the smallest of the non-empty set  $\{m : \sigma(n) \neq \beta(n)\}$ . In what follows elements of  $\mathbf{R}_2$  are repeatedly identified by using R10.2.6. If  $m = 1$ , let  $\mathcal{F}$  in  $\mathbf{R}_2$  be associated with the sequence  $\{x_n\}$  where  $x_n = 1$  for all  $n$  and let  $\mathcal{G}$  in  $\mathbf{R}_2$  be associated with the sequence  $\{y_n\}$  where  $y_n = 2$  for all  $n$ . By R19.Add.2  $\mathcal{F} <_2 \mathcal{G}$ . Since  $\{x_n\}$  is the  $\sigma$ -sequence of  $\psi_\sigma(\mathcal{F})$  and  $\sigma(1) = 0$ , by the definition of  $\sigma$ -sequence  $\psi_\sigma(\mathcal{F}) \in l([0, 1])$ . Similarly,  $\psi_\sigma(\mathcal{G}) \in r([0, 1])$ . Thus  $\psi_\sigma(\mathcal{F}) < \psi_\sigma(\mathcal{G})$ . If  $m > 1$ , let  $\mathcal{F}$  in  $\mathbf{R}_2$  be associated with the sequence  $\{x_n\}$  where  $x_n = 1$  for all  $n$  and let  $\mathcal{G}$  in  $\mathbf{R}_2$  be associated with the sequence  $\{y_n\}$  where  $y_n = 1$  for  $n \leq m - 1$  and  $y_n = 1 + 2^{m-1}$  for  $n \geq m$ . Again by R19.Add.2  $\mathcal{F} <_2 \mathcal{G}$  and by definition of  $\psi_\sigma$ ,  $\{x_n\}$  is the  $\sigma$ -sequence of  $\psi_\sigma(\mathcal{F})$  and  $\{y_n\}$  is the  $\sigma$ -sequence of  $\psi_\sigma(\mathcal{G})$ . Since  $x_{m-1} = y_{m-1}$ , by R26.Add.23.ii  $I_{m-1}(\psi_\sigma(\mathcal{F})) = I_{m-1}(\psi_\sigma(\mathcal{G}))$ . Since  $\sigma(m) = 0$ , by definition of the  $\sigma$ -sequence,  $\psi_\sigma(\mathcal{F}) \in l(I_{m-1}(\psi_\sigma(\mathcal{F})))$  and  $\psi_\sigma(\mathcal{G})$  is in  $r(I_{m-1}(\psi_\sigma(\mathcal{F})))$ . Thus  $\psi_\sigma(\mathcal{F}) < \psi_\sigma(\mathcal{G})$  and the conclusion follows.

**Corollary R26.Add.40** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$  be a non-constant map. Then  $\psi_\sigma$  is neither order-preserving nor order-reversing.

Proof: Since  $\sigma$  is non-constant, there exist  $j, k \in \mathbf{N}$  such that  $\sigma(j) = 1$  and  $\sigma(k) = 0$ . By R26.Add.37  $\psi_\sigma$  is not order-preserving and by R26.Add.39  $\psi_\sigma$  is not order-reversing.

The following notation will be used for the rest of this subsection. Let  $n \in \mathbf{N}$ . Recall that  $C_n$  is a finite union of pairwise disjoint subintervals.  $L_n$  denotes the set of left endpoints of intervals in that union and  $R_n$  the set of right endpoints. By construction  $L_n \subseteq L_{n+1}$ ,  $R_n \subseteq R_{n+1}$ , and  $L_n \cap R_n = \emptyset$ . By R26.30  $L_n \cup R_n \subseteq C$ .  $L$  denotes  $\cup_{n=1}^{\infty} L_n$  and  $R$  denotes  $\cup_{n=1}^{\infty} R_n$ . Let  $E = L \cup R$ .

**Lemma R26.Add.41** Let  $x \in C$ . Then

- i)  $x \in L$  if and only if  $x \in l(I_n(x))$  eventually.
- ii)  $x \in R$  if and only if  $x \in r(I_n(x))$  eventually.

Proof: For i): Let  $x \in L$ . Then there is  $m$  such that  $x \in L_m$ , i.e., there is an interval  $J$  in the union making up  $C_m$  such that  $J = [x, b]$ . It will be shown by induction that  $n \geq m$  implies  $x$  is the left endpoint of  $I_n(x)$ . Since  $J = I_m(x)$ , the claim is true for  $m$ . Assume it holds for  $n$ , i.e.,  $I_n(x) = [x, b_n]$ . Clearly  $x$  is the left endpoint of  $l(I_n(x)) = I_{n+1}(x)$  and the claim holds for  $n + 1$ . It follows that  $n \geq m$  implies  $x \in l(I_n(x))$ , i.e.,  $x \in l(I_n(x))$  eventually. Conversely, suppose there is  $m$  such that  $n \geq m$  implies  $x \in l(I_n(x))$ . Write  $I_m(x) = [a, b]$ . By a similar induction,  $a$  is the left endpoint of  $I_n(x)$  for all  $n \geq m$ . By R26.Add.20iii the left endpoints, in this case an eventually constant sequence, converge to  $x$ , i.e.,  $x = a$ . Thus  $x \in L_m \subseteq L$ . Part ii) has a similar proof.

In [6]  $\mathbf{R}_2$  is shown to be a compactification of  $\mathbf{Z}$  with a certain non-discrete topology.

The embedding is  $f_2$ , where, for  $z \in \mathbf{Z}$ ,  $f_2(z)$  has associated sequence  $\{x_n\}$  with  $x_n \equiv z \pmod{2^n}$  for every  $n \in \mathbf{N}$ .

**Lemma R26.Add.42** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ . Then the following are equivalent:

- i) There is  $k \in \mathbf{N}$  such that  $\psi_\sigma(f_2(k)) \in L$ .
- ii)  $\sigma$  is eventually zero.
- iii)  $\psi_\sigma[f_2[\mathbf{N}]] = L$ .

Proof: Assume i). Let  $k \in \mathbf{N}$  such that  $\psi_\sigma(f_2(k)) \in L$  and let  $\{x_n\}$  be the associated sequence of  $f_2(k)$ . As noted above,  $x_n \equiv k \pmod{2^n}$  for all  $n$  and so  $x_n = k$  eventually. By R26.Add.41i  $\psi_\sigma(f_2(k)) \in l(I_n(\psi_\sigma(f_2(k))))$  eventually. Thus there is  $m$  such that  $n \geq m$  implies  $x_n = k$  and  $\psi_\sigma(f_2(k)) \in l(I_n(\psi_\sigma(f_2(k))))$ . By definition of  $\psi_\sigma$  the  $\sigma$ -sequence of  $\psi_\sigma(f_2(k))$  is  $\{x_n\}$ . For  $n \geq m$ , since  $\psi_\sigma(f_2(k))$  is to the left in  $I_n(\psi_\sigma(f_2(k)))$ , by definition of the  $\sigma$ -sequence,  $x_{n+1} = x_n + \sigma(n+1) \cdot 2^n$ . Since  $x_{n+1} = x_n$ ,  $\sigma(n+1) = 0$ . Thus  $\sigma$  is eventually zero. Thus i) implies ii). Now assume ii), i.e.,  $\sigma$  is eventually zero. Let  $j$  be in  $\mathbf{N}$  and let  $\{j_n\}$  be the associated sequence of  $f_2(j)$ . Thus there is  $m$  such that  $n \geq m$  implies  $\sigma(n) = 0$  and  $j_n = j$ . By the definition of  $\psi_\sigma$ ,  $\{j_n\}$  is also the  $\sigma$ -sequence of  $\psi_\sigma(f_2(j))$ . For  $n \geq m$ ,  $j_{n+1} = j = j_n$ . Since  $\sigma(n+1) = 0$ , by the definition of  $\sigma$ -sequence,  $\psi_\sigma(f_2(j))$  must be in  $l(I_n(\psi_\sigma(f_2(j))))$ . By R26.Add.41i  $\psi_\sigma(f_2(j)) \in L$ . Thus so far  $\psi_\sigma[f_2[\mathbf{N}]] \subseteq L$ . Now let  $t \in L$  with  $\{t_n\}$  the  $\sigma$ -sequence of  $t$ . By R26.Add.41i there is  $l \in \mathbf{N}$  such that  $n \geq l$  implies  $t \in l(I_n(t))$  and  $\sigma(n) = 0$ . For  $n \geq l$ , by the definition of the  $\sigma$ -sequence,  $t_{n+1} = t_n + \sigma(n+1) \cdot 2^n = t_n$ . It follows by an easy induction that  $t_n = t_l$  for all  $n \geq l$ . By R26.Add.22iii  $t_n \equiv t_l \pmod{2^n}$  for all  $n \leq l$  and so  $t_n \equiv t_l \pmod{2^n}$  for all  $n$  in  $\mathbf{N}$ . By definition  $\{t_n\}$  is the associated sequence of  $\psi_\sigma^{-1}(t)$  and as noted above it is also the associated sequence of  $f_2(t_l)$ . By R10.2.4  $\psi_\sigma^{-1}(t) = f_2(t_l)$ . Thus  $t \in \psi_\sigma[f_2[\mathbf{N}]]$  so that  $L \subseteq \psi_\sigma[f_2[\mathbf{N}]]$  and iii) holds. Clearly iii) implies i).

**Lemma R26.Add.43** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ . Then the following are equivalent:

- i) There is  $k \in \mathbf{N}$  such that  $\psi_\sigma(f_2(k)) \in R$ .
- ii)  $\sigma$  is eventually one.
- iii)  $\psi_\sigma[f_2[\mathbf{N}]] = R$ .

Proof: The proof is similar to that of the previous result, with R26.Add.41ii being the key. Assume i). Let  $k \in \mathbf{N}$  such that  $\psi_\sigma(f_2(k)) \in R$  and let  $\{x_n\}$  be the associated sequence of  $f_2(k)$ . As noted above,  $x_n \equiv k \pmod{2^n}$  for all  $n$  and so  $x_n = k$  eventually. By R26.Add.41ii  $\psi_\sigma(f_2(k)) \in r(I_n(\psi_\sigma(f_2(k))))$  eventually. Thus there is  $m$  such that  $n \geq m$  implies  $x_n = k$  and  $\psi_\sigma(f_2(k)) \in r(I_n(\psi_\sigma(f_2(k))))$ . By definition of  $\psi_\sigma$  the  $\sigma$ -sequence of  $\psi_\sigma(f_2(k))$  is  $\{x_n\}$ . For  $n \geq m$ , since  $\psi_\sigma(f_2(k))$  is to the right in  $I_n(\psi_\sigma(f_2(k)))$ , by definition of the  $\sigma$ -sequence,  $x_{n+1} = x_n + (1 - \sigma(n+1)) \cdot 2^n$ . Since  $x_{n+1} = x_n$ ,  $\sigma(n+1) = 1$ . Thus  $\sigma$  is eventually one and so i) implies ii). Now assume ii), i.e.,  $\sigma$  is eventually one. Let  $j$  be in  $\mathbf{N}$  and let  $\{j_n\}$  be the associated sequence of  $f_2(j)$ . Thus there is  $m$  such that  $n \geq m$  implies  $\sigma(n) = 1$  and  $j_n = j$ . By the definition of  $\psi_\sigma$ ,  $\{j_n\}$  is also the  $\sigma$ -sequence of  $\psi_\sigma(f_2(j))$ . For  $n \geq m$ ,  $j_{n+1} = j = j_n$ . Since  $\sigma(n+1) = 1$ , by the definition of  $\sigma$ -sequence,  $\psi_\sigma(f_2(j))$  must be in  $r(I_n(\psi_\sigma(f_2(j))))$ . By R26.Add.41ii  $\psi_\sigma(f_2(j)) \in R$ . Thus so far  $\psi_\sigma[f_2[\mathbf{N}]] \subseteq R$ . Now let  $t \in R$  with  $\{t_n\}$  the  $\sigma$ -sequence of  $t$ . By R26.Add.41i there is  $l \in \mathbf{N}$  such that  $n \geq l$  implies  $t \in r(I_n(t))$  and  $\sigma(n) = 1$ . For  $n \geq l$ , by the definition of the  $\sigma$ -sequence,  $t_{n+1} = t_n + (1 - \sigma(n+1)) \cdot 2^n = t_n$ . It follows by an easy induction that  $t_n = t_l$  for all  $n \geq l$ . By R26.Add.22iii  $t_n \equiv t_l \pmod{2^n}$  for all  $n \leq l$  and so  $t_n \equiv t_l \pmod{2^n}$  for every

$n$  in  $\mathbf{N}$ . By definition  $\{t_n\}$  is the associated sequence of  $\psi^{-1}(t)$  and as noted above it is also the associated sequence of  $f_2(t_l)$ . By R10.2.4  $\psi^{-1}(t) = f_2(t_l)$ . Thus  $R \subseteq \psi_\sigma[f_2[\mathbf{N}]]$  so that iii) holds. Clearly iii) implies i).

**Lemma R26.Add.44** Let  $k \in \mathbf{N} \cup \{0\}$  and let  $\{x_n\}$  be the associated sequence of  $f_2(-k)$ . Then  $x_n = 2^n - k$  eventually.

Proof: As noted above,  $x_n \equiv -k \pmod{2^n}$  and so  $x_n \equiv 2^n - k \pmod{2^n}$  for every  $n$ . Pick  $m$  such that  $2^m > k$ . For  $n \geq m$ ,  $2^n \geq 2^m$  so that  $2^n > k$ . Thus  $1 \leq 2^n - k \leq 2^n$ , i.e.,  $2^n - k \in \{1, \dots, 2^n\}$ . Distinct elements of that set are not equivalent mod  $2^n$  and  $x_n$  must be in that set. Thus  $x_n = 2^n - k$ .

**Lemma R26.Add.45** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$  and let  $S$  be the set of non-positive integers. Then the following are equivalent:

- i) There is  $k \in \mathbf{N} \cup \{0\}$  such that  $\psi_\sigma(f_2(-k)) \in R$ .
- ii)  $\sigma$  is eventually zero.
- iii)  $\psi_\sigma[f_2[S]] = R$ .

Proof: Assume  $k \in \mathbf{N} \cup \{0\}$  with  $\psi_\sigma(f_2(-k)) \in R$ , and let  $\{x_n\}$  be the  $\sigma$ -sequence of  $\psi_\sigma(f_2(-k))$ . By definition of  $\psi_\sigma^{-1}$ ,  $\{x_n\}$  is also the associated sequence of  $f_2(-k)$ . By hypothesis, R26.Add.41ii, and R26.Add.44 there is  $m$  such that  $n \geq m$  implies  $\psi_\sigma(f_2(-k))$  is in  $r(I_n(\psi_\sigma(f_2(-k))))$  and  $x_n = 2^n - k$ . Let  $n \geq m$ . Since  $\psi_\sigma(f_2(-k)) \in r(I_n(\psi_\sigma(f_2(k))))$ , by definition of the  $\sigma$ -sequence,  $x_{n+1} = x_n + (1 - \sigma(n+1)) \cdot 2^n$ , i.e.,  $2^{n+1} - k$  must equal  $2^n - k + (1 - \sigma(n+1)) \cdot 2^n$ . Thus  $\sigma(n+1)$  must be 0 and so i) implies ii). Now assume  $\sigma$  is eventually zero. Let  $-k$  be in  $S$ , where  $k \in \mathbf{N} \cup \{0\}$ , and let  $\{x_n\}$  be the associated sequence of  $f_2(-k)$ . By the assumption for this part and R26.Add.44 there is  $m$  such that  $n \geq m$  implies  $\sigma(n) = 0$  and  $x_n = 2^n - k$ . Note that  $n \geq m$  implies  $x_{n+1} = 2^{n+1} - k = 2^n + 2^n - k = x_n + 2^n$ . Again,  $\{x_n\}$  is also the  $\sigma$ -sequence of  $\psi_\sigma(f_2(-k))$ . For  $n \geq m$ ,  $\sigma(n+1) = 0$  and  $x_{n+1} \neq x_n$  imply, by definition of the  $\sigma$ -sequence, that  $x_{n+1} = x_n + (1 - \sigma(n+1)) \cdot 2^n$ , i.e.,  $\psi_\sigma(f_2(-k))$  must be in  $r(I_n(\psi_\sigma(f_2(-k))))$ . By R26.Add.41ii  $\psi_\sigma(f_2(-k)) \in R$ . Thus  $\psi_\sigma[f_2[S]] \subseteq R$ . Now let  $t \in R$  have  $\sigma$ -sequence  $\{t_n\}$ . By the assumption and R26.Add.41ii, there is  $m$  such that  $n \geq m$  implies  $\sigma(n) = 0$  and  $t \in r(I_n(t))$ . Thus, for  $n \geq m$ ,  $t_{n+1} = t_n + (1 - \sigma(n+1)) \cdot 2^n = t_n + 2^n$ . Let  $k = 2^m - t_m$ . Since  $t_m \in \{1, \dots, 2^m\}$ ,  $0 \leq k \leq 2^m - 1$  and so  $-k \in S$ . A routine induction shows that  $t_n = 2^n - k$  and so  $t_n \equiv -k \pmod{2^n}$  for all  $n \geq m$ . For  $n \leq m$ , by R26.Add.22iii,  $t_n \equiv t_m = 2^m - k \pmod{2^n}$  and so  $t_n \equiv -k \pmod{2^n}$ . Thus  $t_n \equiv -k \pmod{2^n}$  for all  $n$ . By definition  $\{t_n\}$  is the associated sequence of  $\psi_\sigma^{-1}(t)$ . Since  $\psi_\sigma^{-1}(t)$  and  $f_2(-k)$  have the same associated sequence, by R10.2.4  $\psi_\sigma^{-1}(t) = f_2(-k)$ , i.e.,  $t = \psi_\sigma(f_2(k))$ . Thus  $R \subseteq \psi_\sigma[f_2[S]]$  so that iii) holds. Clearly iii) implies i).

**Lemma R26.Add.46** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$  and let  $S$  be the set of non-positive integers. Then the following are equivalent:

- i) There is  $k \in \mathbf{N} \cup \{0\}$  such that  $\psi_\sigma(f_2(-k)) \in L$ .
- ii)  $\sigma$  is eventually one.
- iii)  $\psi_\sigma[f_2[S]] = L$ .

Proof: Assume  $k \in \mathbf{N} \cup \{0\}$  with  $\psi_\sigma(f_2(-k)) \in L$ , and let  $\{x_n\}$  be the  $\sigma$ -sequence of  $\psi_\sigma(f_2(-k))$ . By definition of  $\psi_\sigma^{-1}$ ,  $\{x_n\}$  is also the associated sequence of  $f_2(-k)$ . By hypothesis, R26.Add.41i, and R26.Add.44 there is  $m$  such that  $n \geq m$  implies  $\psi_\sigma(f_2(-k))$  is in  $l(I_n(\psi_\sigma(f_2(-k))))$  and  $x_n = 2^n - k$ . Let  $n \geq m$ . Since  $\psi_\sigma(f_2(-k)) \in l(I_n(\psi_\sigma(f_2(k))))$ ,

by definition of the  $\sigma$ -sequence,  $x_{n+1} = x_n + \sigma(n+1) \cdot 2^n$ , i.e.,  $2^{n+1} - k$  must equal  $2^n - k + \sigma(n+1) \cdot 2^n$ . Thus  $\sigma(n+1)$  must be 1 and so i) implies ii). Now assume  $\sigma$  is eventually one. Let  $-k$  be in  $S$ , where  $k \in \mathbf{N} \cup \{0\}$ , and let  $\{x_n\}$  be the associated sequence of  $f_2(-k)$ . By the assumption for this part and R26.Add.44 there is  $m$  such that  $n \geq m$  implies  $\sigma(n) = 1$  and  $x_n = 2^n - k$ . Note that  $n \geq m$  implies  $x_{n+1} = 2^{n+1} - k = x_n + 2^n$ . Again,  $\{x_n\}$  is also the  $\sigma$ -sequence of  $\psi_\sigma(f_2(-k))$ . For  $n \geq m$ ,  $\sigma(n+1) = 1$  and  $x_{n+1} \neq x_n$  imply, by definition of the  $\sigma$ -sequence, that  $x_{n+1} = x_n + \sigma(n+1) \cdot 2^n$ , i.e.,  $\psi_\sigma(f_2(-k))$  must be in  $l(I_n(\psi_\sigma(f_2(-k))))$ . By R26.Add.41i  $\psi_\sigma(f_2(-k)) \in L$ . Thus  $\psi_\sigma[f_2[S]] \subseteq L$ . Now let  $t \in L$  have  $\sigma$ -sequence  $\{t_n\}$ . By the assumption and R26.Add.41i, there is  $m$  such that  $n \geq m$  implies  $\sigma(n) = 1$  and  $t \in l(I_n(t))$ . Thus, for  $n \geq m$ ,  $t_{n+1} = t_n + \sigma(n+1) \cdot 2^n = t_n + 2^n$ . Let  $k = 2^m - t_m$ . Since  $t_m \in \{1, \dots, 2^m\}$ ,  $0 \leq k \leq 2^m - 1$  and so  $-k \in S$ . A routine induction shows that  $t_n = 2^n - k$  and so  $t_n \equiv -k \pmod{2^n}$  for all  $n \geq m$ . For  $n \leq m$ , by R26.Add.22iii,  $t_n \equiv t_m = 2^m - k \pmod{2^n}$  and so  $t_n \equiv -k \pmod{2^n}$ . Thus  $t_n \equiv -k \pmod{2^n}$  for all  $n$ . By definition  $\{t_n\}$  is the associated sequence of  $\psi_\sigma^{-1}(t)$ . Since  $\psi_\sigma^{-1}(t)$  and  $f_2(-k)$  have the same associated sequence, by R10.2.4  $\psi_\sigma^{-1}(t) = f_2(-k)$ , i.e.,  $t = \psi_\sigma(f_2(k))$ . Thus  $L \subseteq \psi_\sigma[f_2[S]]$  so that iii) holds. Clearly iii) implies i).

**Corollary R26.Add.47** Let  $\sigma : \mathbf{N} \rightarrow \{0, 1\}$ . The following are equivalent:

- i)  $\psi_\sigma[\mathbf{Z}] = E$ .
- ii)  $\psi_\sigma[\mathbf{Z}] \cap E \neq \emptyset$ .
- iii)  $\sigma$  is eventually constant.

Proof: Clearly i) implies ii). Assume ii) and let  $z \in \mathbf{Z}$  with  $\psi_\sigma(z) \in E$ . If  $\psi_\sigma(z) \in L$ , by R26.Add.46  $\sigma$  is eventually one if  $z$  is non-positive, and by R26.Add.42  $\sigma$  is eventually zero if  $z$  is positive. If  $\psi_\sigma(z) \in R$ , R26.Add.45 and R26.Add.43 can be used similarly. Thus ii) implies iii). Now assume iii). If  $\sigma$  is eventually zero, i) follows from R26.Add.42 and R26.Add.45. If  $\sigma$  is eventually one, i) follows from R26.Add.43 and R26.Add.46.

**Corollary R26.Add.48** Let  $A \subseteq \mathbf{N}$ . Let  $\alpha : \mathbf{N} \rightarrow \{0, 1\}$  by  $\alpha(n) = 1$  if  $n \in A$  and  $\alpha(n) = 0$  if  $n \notin A$ . Then  $A$  is finite or co-finite if and only if  $\psi_\alpha[\mathbf{Z}] \cap E \neq \emptyset$ .

Proof: Clearly  $A$  is finite if and only if the characteristic function  $\alpha$  is eventually zero. Likewise  $A$  is co-finite if and only if  $\alpha$  is eventually one. Combining these observations with the previous result yields this corollary.