

Disjoint Unions of Uniform Spaces

Most of this subsection is undoubtedly known but it is included here because I have no reference. The approach follows the pattern used for the disjoint union of topological spaces in [4]. As in R15.1.12, given an indexed family of sets $\{X_\alpha : \alpha \in \Delta\}$ with $\Delta \neq \emptyset$, the disjoint union is defined to be $\cup\{X_\alpha \times \{\alpha\} : \alpha \in \Delta\}$ and is denoted $\coprod\{X_\alpha : \alpha \in \Delta\}$.

Definition R24.1 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. For $U \in \mathcal{U}_\alpha$, U^* is defined to be $\{((x, \alpha), (y, \alpha)) : (x, y) \in U\}$.

The uniformity of the next definition appeared without the notation in R15.1.17.

Definition R24.2 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Let $W \subseteq D \times D$, where for notational convenience D denotes $\coprod\{X_\alpha : \alpha \in \Delta\}$. W is in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ if and only if there is $c \in \prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ such that $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$.

The last definition contains two ambiguities which, it is hoped, will always be resolvable by context. (The alternative would be more notational complexity.) First, the definition intends to specify a set denoted $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$, which might be referred to as the coproduct uniformity, but the same notation might also indicate the disjoint union of the sets $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Secondly, the point c is in the product of the sets $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$, not the product uniformity.

The first proposition justifies calling $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ a uniformity and relates it to the disjoint union of the associated topological spaces $\{(X_\alpha, \tau(\mathcal{U}_\alpha)) : \alpha \in \Delta\}$, the topology of which is denoted $\coprod\{\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

Proposition R24.3 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Then

- i) $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is a uniformity on $\coprod\{X_\alpha : \alpha \in \Delta\}$.
- ii) $\tau(\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}) = \coprod\{\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

Proof: The routine proof of i) will be illustrated by a verification of the triangle inequality. Let W be in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ and let c be in the product $\prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ such that $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$. Define d in the product as follows: For $\alpha \in \Delta$ let $d(\alpha)$ be a symmetric entourage in \mathcal{U}_α such that $d(\alpha) \circ d(\alpha) \subseteq c(\alpha)$. Let $V = \cup\{d(\alpha)^* : \alpha \in \Delta\}$, which by definition is in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. It is easy to check that $V \circ V \subseteq W$ as required. For ii) first let $p \in O \in \tau(\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$. There is W in the uniformity with $W[p] \subseteq O$ and by definition there is $c \in \prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ such that $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$. There is $\delta \in \Delta$ and $x \in X_\delta$ such that $p = (x, \delta)$. There is $G \in \tau(\mathcal{U}_\delta)$ such that $x \in G \subseteq c(\delta)[x]$. Then $G \times \{\delta\}$ is in $\coprod\{\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$ and $p \in G \times \{\delta\} \subseteq c(\delta)^*[p] \subseteq W[p]$ and so O is in $\coprod\{\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$. For the converse containment, it is sufficient to show that every basic open set in $\coprod\{\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$ is also in $\tau(\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$. To that end, let $\delta \in \Delta$ and let $x \in G$, where $G \in \tau(\mathcal{U}_\delta)$. There is $U \in \mathcal{U}_\delta$ such that $U[x] \subseteq G$. Define c by $c(\delta) = U$ and $c(\alpha) = X_\alpha \times X_\alpha$ if $\alpha \neq \delta$. By definition $W = \cup\{c(\alpha)^* : \alpha \in \Delta\}$ is in the coproduct uniformity and it is easy to check that $W[(x, \delta)] \subseteq G \times \{\delta\}$. Thus the basic open set $G \times \{\delta\}$ is in $\tau(\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$ as required.

Proposition R24.4 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is separated if and only if \mathcal{U}_α is separated for every $\alpha \in \Delta$.

Proof: Assume $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is separated, let $\delta \in \Delta$, and let $x \neq y$ be in X_δ . Pick W in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ such that $((x, \delta), (y, \delta))$ is not in W . Pick c such that W contains $\cup\{c(\alpha)^* : \alpha \in \Delta\}$. By definition $((x, \delta), (y, \delta))$ not in $c(\delta)^*$ implies $(x, y) \notin c(\delta)$. Thus \mathcal{U}_δ

is separated. Now assume \mathcal{U}_α is separated for every $\alpha \in \Delta$ and let $(x, \beta) \neq (y, \gamma)$ be in $\coprod\{X_\alpha : \alpha \in \Delta\}$. If $\beta \neq \gamma$, let $c(\alpha) = X_\alpha \times X_\alpha$ for every $\alpha \in \Delta$. Clearly $((x, \beta), (y, \gamma))$ is not in $\cup\{c(\alpha)^* : \alpha \in \Delta\}$. If $\beta = \gamma$, $x \neq y$. Pick U in \mathcal{U}_β such that $(x, y) \notin U$ and let $c(\beta) = U$ and $c(\alpha) = X_\alpha \times X_\alpha$ for $\alpha \neq \beta$. Then $((x, \beta), (y, \beta))$ is not in $c(\beta)^*$ and so not in $\cup\{c(\alpha)^* : \alpha \in \Delta\}$. Thus $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is separated.

Lemma R24.5 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Let $S : D \rightarrow \coprod\{X_\alpha : \alpha \in \Delta\}$ be a $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ -Cauchy net. Then there is $\delta \in \Delta$ such that S is eventually in $X_\delta \times \{\delta\}$.

Proof: Let $c(\alpha) = X_\alpha \times X_\alpha$ for every $\alpha \in \Delta$. By definition $W = \cup\{c(\alpha)^* : \alpha \in \Delta\}$ is in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Since S is Cauchy, there is $d_0 \in D$ such that $d, e \geq d_0$ implies $(S(d), S(e)) \in W$. Let $s(d_0) = (x_0, \delta)$. For $d \geq d_0$, there is $\alpha \in \Delta$ such that $(S(d), S(d_0))$ is in $c(\alpha)^*$, which implies that $\alpha = \delta$ and $S(d)$ is in $X_\delta \times \{\delta\}$.

Proposition R24.6 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is complete if and only if \mathcal{U}_α is complete for every $\alpha \in \Delta$.

Proof: First assume \mathcal{U}_α is complete for every $\alpha \in \Delta$ and let $S : D \rightarrow \coprod\{X_\alpha : \alpha \in \Delta\}$ be a $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ -Cauchy net. By the lemma there is δ and $d_0 \in D$ such that $d \geq d_0$ implies $S(d) \in X_\delta \times \{\delta\}$. Let D_1 be the directed set $\{d \in D : d \geq d_0\}$ with the order inherited from D . Define $T : D_1 \rightarrow X_\delta$ by $T(d) = x$ where $S(d) = (x, \delta)$. For $U \in \mathcal{U}_\delta$, let $c(\delta) = U$ and, for $\alpha \neq \delta$, $c(\alpha) = X_\alpha \times X_\alpha$. Using the entourage $\cup c(\alpha)^*$ and the fact that S is Cauchy, one easily sees that T is \mathcal{U}_δ -Cauchy. By assumption T has a limit $x_0 \in X_\delta$. It is claimed that S has (x_0, δ) as a limit. Let W be in the coproduct uniformity and pick c such that $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$. There is $d_1 \geq d_0$ such $d \geq d_1$ implies $T(d) \in c(\delta)[x_0]$. Thus $d \geq d_1$ implies $S(d) \in c(\delta)^*[(x_0, \delta)] \subseteq W[(x_0, \delta)]$ and so the claim holds. Conversely assume $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is complete, let $\beta \in \Delta$, and let $S : D \rightarrow X_\beta$ be \mathcal{U}_β -Cauchy. Define $T : D \rightarrow \coprod\{X_\alpha : \alpha \in \Delta\}$ by $T(d) = (S(d), \beta)$. Given W in the coproduct uniformity, pick c with $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$. There is $d_0 \in D$ such that $d, e \geq d_0$ implies $(S(d), S(e)) \in c(\beta)$. It follows that $d, e \geq d_0$ implies $(T(d), T(e)) \in c(\beta)^* \subseteq W$ and so T is $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ -Cauchy. By assumption T has a limit (x_0, γ) . First $\gamma = \beta$ (and so $x_0 \in X_\beta$) will be verified. Define c_1 by $c_1(\alpha) = X_\alpha \times X_\alpha$ for every $\alpha \in \Delta$. There is $d_1 \in D$ such that $d \geq d_1$ implies $T(d) \in \cup\{c_1(\alpha)^* : \alpha \in \Delta\}[(x_0, \gamma)] = c_1(\gamma)^*[(x_0, \gamma)]$. Since $T(d) = (S(d), \beta)$, the claim holds. Finally it will be shown that S has x_0 as a limit. Let U be in \mathcal{U}_β . Define $e(\beta) = U$ and $e(\alpha) = X_\alpha \times X_\alpha$ for $\alpha \neq \beta$. $W = \cup\{e(\alpha)^* : \alpha \in \Delta\}$ is in the coproduct uniformity and so there is $d_2 \in D$ such that $d \geq d_2$ implies $T(d) \in W[(x_0, \beta)]$. Since $W[(x_0, \beta)] = e(\beta)^*[(x_0, \beta)] = U^*[(x_0, \beta)]$. It follows that $d \geq d_2$ implies $S(d) \in U[x_0]$, i.e., S has x_0 as a limit. Thus \mathcal{U}_β is complete.

Proposition R24.7 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of non-empty uniform spaces. $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is totally bounded if and only if Δ is finite and \mathcal{U}_α is totally bounded for every $\alpha \in \Delta$.

Proof: First assume $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is totally bounded and suppose Δ is infinite. Define $d(\alpha) = X_\alpha \times X_\alpha$ for every $\alpha \in \Delta$ and let $W = \cup\{d(\alpha)^* : \alpha \in \Delta\}$, which is in the coproduct uniformity. For any finite set of points $\{(x_i, \alpha_i) : 1 \leq i \leq n\}$ in $\coprod\{X_\alpha : \alpha \in \Delta\}$, pick $\delta \in \Delta - \{\alpha_1, \dots, \alpha_n\}$. Since $W[(x_i, \alpha_i)] = X_{\alpha_i} \times \{\alpha_i\}$, no point of $X_\delta \times \{\delta\}$ is in $\cup\{W[(x_i, \alpha_i)] : 1 \leq i \leq n\}$, which contradicts the total-boundedness. Next fix $\gamma \in \Delta$ and let $U \in \mathcal{U}_\gamma$. Define $e(\gamma) = U$ and $e(\alpha) = X_\alpha \times X_\alpha$ if $\alpha \neq \gamma$. Let $V = \cup\{e(\alpha)^* : \alpha \in \Delta\}$.

By hypothesis, $\coprod\{X_\alpha : \alpha \in \Delta\}$ contains a finite set F such that $V[F] = \coprod\{X_\alpha : \alpha \in \Delta\}$. Since $V[(x, \alpha)] = e(\alpha)^*[(x, \alpha)] = e(\alpha)[x] \times \{\alpha\}$ and $X_\gamma \neq \emptyset$, $S = \{(x, \gamma) \in F\}$, which is finite, is clearly non-empty and $U[S] = X_\gamma$. Thus \mathcal{U}_γ is totally bounded. For the converse it is sufficient to consider a basic entourage, say $W = \cup\{c(\alpha)^* : \alpha \in \Delta\}$. For each α there is a finite set F_α such that $c(\alpha)[F_\alpha] = X_\alpha$. Since Δ is finite, the set $F = \cup\{F_\alpha \times \{\alpha\} : \alpha \in \Delta\}$ is finite and it follows easily that $W[F] = \coprod\{X_\alpha : \alpha \in \Delta\}$.

A less old-fashioned approach to the disjoint union involves a universal property of morphisms, which allows an easy way to recognize a space as isomorphic to a disjoint union. What follows is a specialization of the categorical presentation of coproducts (e.g., see pp.62-64 of [1]). Obviously similar modifications could be made for disjoint unions of sets or topological spaces.

Definition R24.8 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. For each $\gamma \in \Delta$, $i_\gamma : X_\gamma \rightarrow \coprod\{X_\alpha : \alpha \in \Delta\}$ is defined by $i_\gamma(x) = (x, \gamma)$.

Lemma R24.9 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces and let $\gamma \in \Delta$. Then $i_\gamma : (X_\gamma, \mathcal{U}_\gamma) \rightarrow (\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$ is uniformly continuous.

Proof: Let $W \in \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Then there is a collection $\{U_\alpha : \alpha \in \Delta\}$ with each $U_\alpha \in \mathcal{U}_\alpha$ such that $\cup\{U_\alpha^* : \alpha \in \Delta\} \subseteq W$. Clearly $(i_\gamma \times i_\gamma)^{-1}[U_\gamma^*] = U_\gamma$. Thus $U_\gamma \subseteq (i_\gamma \times i_\gamma)^{-1}[W]$ and so i_γ is uniformly continuous.

Lemma R24.10 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Let (Z, \mathcal{W}) be a uniform space and assume for each $\gamma \in \Delta$ there is $h_\gamma : (X_\gamma, \mathcal{U}_\gamma) \rightarrow (Z, \mathcal{W})$ uniformly continuous. There is a unique $H : (\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}) \rightarrow (Z, \mathcal{W})$ uniformly continuous such that, for every $\alpha \in \Delta$, $h_\alpha = H \circ i_\alpha$.

Proof: Define $H : \coprod\{X_\alpha : \alpha \in \Delta\} \rightarrow Z$ by $H(t, \gamma) = h_\gamma(t)$. Clearly $H \circ i_\alpha = h_\alpha$ for all $\alpha \in \Delta$. For $W \in \mathcal{W}$, $(H \times H)^{-1}[W]$ contains $\cup\{((h_\alpha \times h_\alpha)^{-1}[W])^* : \alpha \in \Delta\}$, which is in the coproduct uniformity. Thus H is uniformly continuous. Lastly suppose G is uniformly continuous and $G \circ i_\alpha = h_\alpha$ for all α . For $(t, \gamma) \in \coprod\{X_\alpha : \alpha \in \Delta\}$, $G((t, \gamma)) = G \circ i_\gamma(t) = h_\gamma(t) = H \circ i_\gamma(t) = H((t, \gamma))$. Thus $G = H$.

The last three items describe the universal property mentioned above. The next proposition shows that it determines the disjoint union up to unimorphism. The proof is a version of a standard categorical argument.

Proposition R24.11 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Let (Y, \mathcal{V}) be a uniform space and $\{f_\alpha : X_\alpha \rightarrow Y : \alpha \in \Delta\}$ be a collection of uniformly continuous maps. Assume that for any uniform space (Z, \mathcal{W}) and for any collection $\{h_\alpha : X_\alpha \rightarrow Z : \alpha \in \Delta\}$ of uniformly continuous maps there is a unique uniformly continuous map $G : Y \rightarrow Z$ such that $G \circ f_\alpha = h_\alpha$. Then $(\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$ is unimorphic to (Y, \mathcal{V}) by a unimorphism mapping (x, α) to $f_\alpha(x)$.

Proof: Apply the assumption to the collection $\{i_\alpha\}$ to obtain $G : Y \rightarrow \coprod\{X_\alpha : \alpha \in \Delta\}$ uniformly continuous with $G \circ f_\alpha = i_\alpha$ for all α . Apply R24.9 to the collection $\{f_\alpha\}$ to obtain $H : \coprod\{X_\alpha : \alpha \in \Delta\} \rightarrow Y$ uniformly continuous with $H \circ i_\alpha = f_\alpha$ for all α . Then $G \circ H \circ i_\alpha = i_\alpha$ for all α and by the uniqueness in R24.10 $G \circ H$ must be the identity map on $\coprod\{X_\alpha : \alpha \in \Delta\}$. Also $H \circ G \circ f_\alpha = f_\alpha$ for all α and by the uniqueness in the assumption $H \circ G$ must be the identity map on Y . Thus G and H are inverses of each other and so the spaces are unimorphic as required. The equation $H \circ i_\alpha = f_\alpha$ says that the unimorphism H maps (x, α) to $f_\alpha(x)$.

The previous proposition can be used to see that certain constructions used previously lead to spaces unimorphic to a disjoint union.

Corollary R24.12 Let (X, \mathcal{U}) be a uniform space and let, E be an equivalence relation on X with finitely many equivalence classes $\{C_1, \dots, C_n\}$. Let \mathcal{U}_j be the subspace uniformity from \mathcal{U} on C_j . Then $(\coprod_{j=1}^n C_j, \coprod_{j=1}^n \mathcal{U}_j)$ is unimorphic to $(X, \mathcal{U} \vee \mathcal{U}_E)$ by a unimorphism mapping (x, j) to x .

Proof: For each j , let ι_j be the inclusion map from C_j to X . The fact that ι_j is uniformly continuous from (C_j, \mathcal{U}_j) to $(X, \mathcal{U} \vee \mathcal{U}_E)$ follows immediately from the hypothesis and $(\iota_j \times \iota_j)^{-1}[U \cap E] = U \cap (C_j \times C_j)$. Now let (Z, \mathcal{W}) be a uniform space and, for each j , let $f_j : (C_j, \mathcal{U}_j) \rightarrow (Z, \mathcal{W})$ be uniformly continuous. Define $F : X \rightarrow Z$ by $F(x) = f_j(x)$ if $x \in C_j$. Clearly $F \circ \iota_j = f_j$. For each j pick $U_j \in \mathcal{U}_j$ such that $(f_j \times f_j)^{-1}[W] = U_j \cap (C_j \times C_j)$. It is easy to check that $(\cap_{j=1}^n U_j) \cap E \subseteq (F \times F)^{-1}[W]$ and so F is uniformly continuous from $(X, \mathcal{U} \vee \mathcal{U}_E)$ to (Z, \mathcal{W}) . The uniqueness of F is immediate from the functional equations. Since each ι_j is inclusion, by R24.11 the conclusion holds.

In a similar way, the uniformity constructed in R23.3.5 can be recognized as unimorphic to a disjoint union. The next two corollaries also relate to prior results. For \mathcal{A} a collection of subsets of a set X , the equivalence relation associated with \mathcal{A} , denoted $E(\mathcal{A})$, was defined in R23.4.8. The sets $\mathcal{C}(\mathcal{A})$ and B_p were defined in R22.2.2.

Corollary R24.13 Let (X, \mathcal{U}) be a uniform space and let \mathcal{A} be a finite collection of subsets of X . For each $p \in \mathcal{C}(\mathcal{A})$, let \mathcal{U}_p be the subspace uniformity from \mathcal{U} on B_p . Then $(\coprod\{B_p : p \in \mathcal{C}(\mathcal{A})\}, \coprod\{\mathcal{U}_p : p \in \mathcal{C}(\mathcal{A})\})$ is unimorphic to $(X, \mathcal{U}_e(\mathcal{A}))$ by a unimorphism mapping (x, p) to x .

Proof: Since \mathcal{A} is finite, by R23.4.10 $\mathcal{U}_e(\mathcal{A}) = \mathcal{U} \vee \mathcal{U}_{E(\mathcal{A})}$. $E(\mathcal{A})$ has finitely many equivalence classes, which are the non-empty B_p . The conclusion now follows from R24.12.

The next is equivalent to the converse of R23.3.8.

Corollary R24.14 Let (X, \mathcal{U}) be a separated, totally bounded uniform space and let \mathcal{A} be a finite collection of subsets of X . For each $p \in \mathcal{C}(\mathcal{A})$ assume the subspace uniformity on B_p corresponds to the Stone-Ćech compactification of B_p with the subspace topology from $\tau(\mathcal{U})$. Then $\mathcal{U}_e(\mathcal{A})$ corresponds to the Stone-Ćech compactification of $(X, \tau(\mathcal{U}_e(\mathcal{A})))$.

Proof: Since the Stone-Ćech compactification is characterized by the extendibility of all continuous maps to compact T_2 spaces, let $h : (X, \tau(\mathcal{U}_e(\mathcal{A}))) \rightarrow Z$ be continuous, where Z is a compact T_2 space. By R7.1.1 it is sufficient to show that h is uniformly continuous from $(X, \mathcal{U}_e(\mathcal{A}))$ to Z . For each $p \in \mathcal{C}(\mathcal{A})$, let h_p be h restricted to B_p and let \mathcal{U}_p be the subspace uniformity on B_p from \mathcal{U} . By hypothesis and R7.1.1 h_p is uniformly continuous from (B_p, \mathcal{U}_p) to Z . By R24.10 there is a unique uniformly continuous map H from $(\coprod\{B_p : p \in \mathcal{C}(\mathcal{A})\}, \coprod\{\mathcal{U}_p : p \in \mathcal{C}(\mathcal{A})\})$ to Z such that $h_p = H \circ \iota_p$ for all p . Let ψ be the unimorphism of R24.13, i.e., $\psi(x, p) = x$. It is easy to check that $h = H \circ \psi^{-1}$ and so h is uniformly continuous as required.

The proof of R24.14 can be done directly. With notation as in the last proof, let V be in the unique uniformity for Z . For each p there is U_p in \mathcal{U} such that $U_p \cap B_p = (h_p \times h_p)^{-1}[V]$. The finite intersection $W = \cap\{U_p : p \in \mathcal{C}(\mathcal{A})\}$ is in \mathcal{U} and so $W \cap (\cup\{B_p \times B_p : p \in \mathcal{C}(\mathcal{A})\})$ is in $\mathcal{U}_e(\mathcal{A})$. It is easy to check that $W \cap (\cup\{B_p \times B_p : p \in \mathcal{C}(\mathcal{A})\}) \subseteq (h \times h)^{-1}[V]$, i.e., h is uniformly continuous.

A simple example shows that R24.14 may not hold if \mathcal{A} is infinite: Let X be infinite with the discrete topology, let \mathcal{U} be the uniformity corresponding to the one point compactification of X , and let \mathcal{A} be the collection of all singletons from X . Since $E_{\{x\}} \in \mathcal{U}$ for all x , $\mathcal{U}_e(\mathcal{A}) = \mathcal{U}$, which does not correspond to the Stone-Ćech compactification of X .

Proposition R24.15 Let $\{(X_i, \mathcal{U}_i) : 1 \leq i \leq n\}$ be a finite set of separated, totally bounded uniform spaces. Assume, for each i , that \mathcal{U}_i corresponds to the compactification class $[(Y_i, f_i)]$. Let $g : \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n Y_i$ by $g((x, i)) = (f_i(x), i)$. Then

- i) $(\prod_{i=1}^n X_i, \prod_{i=1}^n \mathcal{U}_i)$ is separated and totally bounded.
- ii) $(\prod_{i=1}^n Y_i, g)$ is a T_2 compactification of $(\prod_{i=1}^n X_i, \tau(\prod_{i=1}^n \mathcal{U}_i))$.
- iii) $\prod_{i=1}^n \mathcal{U}_i$ corresponds to the compactification class $[(\prod_{i=1}^n Y_i, g)]$.

Proof: R24.4 and R24.7 yield i). $\prod_{i=1}^n Y_i$ is compact and T_2 by R15.1.16. Let \mathcal{V}_i be the unique uniformity for Y_i . By R15.1.17 $\prod_{i=1}^n \mathcal{V}_i$ is the unique uniformity for $\prod_{i=1}^n Y_i$. For each j let $\kappa_j : Y_j \rightarrow \prod_{i=1}^n Y_i$ and $\iota_j : X_j \rightarrow \prod_{i=1}^n X_i$ be defined as in R24.8. Apply R24.10 to the uniformly continuous maps $\{\kappa_j \circ f_j\}$ to obtain a unique uniformly continuous map $H : \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n Y_i$ such that $H \circ \iota_i = \kappa_i \circ f_i$ for all i . It is easy to check that $H = g$ and so g is uniformly continuous. Since each f_i is one-to-one, g is also one-to-one. Now let $\cup_{i=1}^n U_i^*$, where $U_i \in \mathcal{U}_i$, be a basic entourage in $\prod_{i=1}^n \mathcal{U}_i$. For each i there is $V_i \in \mathcal{V}_i$ such that $(f_i \times f_i)[U_i] = V_i \cap f_i[X_i] \times f_i[X_i]$. By definition $\cup_{i=1}^n V_i^* \in \prod_{i=1}^n \mathcal{V}_i$ and it is easy to check that $(g \times g)[\cup_{i=1}^n U_i^*] \supseteq (\cup_{i=1}^n V_i^*) \cap (g[\prod_{i=1}^n X_i] \times g[\prod_{i=1}^n X_i])$. Thus g is a unimorphism onto its image. For density of the image, let $\cup_{i=1}^n W_i^*$, where $W_i \in \mathcal{V}_i$, be chosen and let $(y, j) \in \prod_{i=1}^n Y_i$. There is $x \in X_j$ such that $f_j(x) \in W_j[y]$. Since $W_j^*[(y, j)] = W_j[y] \times \{j\}$, $g((x, j)) \in (\cup_{i=1}^n W_i^*)[(y, j)]$, which implies the required density. Thus ii) holds and conclusion iii) follows from R1.6a.

Note that R15.1.19 is a corollary of R24.15 and R15.1.18, since given an equivalence relation E with finitely many classes $\{C_1, \dots, C_n\}$, by R24.12 $(X, \mathcal{U} \vee \mathcal{U}_E)$ is unimorphic to $(\prod_{i=1}^n C_i, \prod_{i=1}^n \mathcal{U}_i)$ where \mathcal{U}_i is the subspace uniformity from \mathcal{U} on C_i .

Lemma R24.16 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty collection of uniform spaces. Assume that for every $\alpha \in \Delta$ there is a collection $\{\mathcal{V}_\gamma : \gamma \in \Gamma_\alpha\}$ of uniformities such that $\mathcal{U}_\alpha = \vee\{\mathcal{V}_\gamma : \gamma \in \Gamma_\alpha\}$. Then $\prod\{\mathcal{U}_\alpha : \alpha \in \Delta\} \supseteq \vee\{\prod\{\mathcal{V}_{p(\alpha)} : \alpha \in \Delta\} : p \in \Pi\{\Gamma_\alpha : \alpha \in \Delta\}\}$, with equality if Δ is finite.

Proof: Let $\mathcal{W} = \vee\{\prod\{\mathcal{V}_{p(\alpha)} : \alpha \in \Delta\} : p \in \Pi\{\Gamma_\alpha : \alpha \in \Delta\}\}$ for notational convenience. Given $p \in \Pi\{\Gamma_\alpha : \alpha \in \Delta\}$, since $\mathcal{V}_{p(\alpha)} \subseteq \mathcal{U}_\alpha$, $\prod\{\mathcal{V}_{p(\alpha)} : \alpha \in \Delta\} \subseteq \prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Thus $\mathcal{W} \subseteq \prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Now assume Δ is finite and let $\cup\{U_\alpha^* : \alpha \in \Delta\}$, where each $U_\alpha \in \mathcal{U}_\alpha$, be a basic entourage in $\prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. For each $\delta \in \Delta$ there is F_δ , a finite subset of Γ_δ , and a set $\{V_t : t \in F_\delta\}$ with each $V_t \in \mathcal{V}_t$ such that $\cap\{V_t : t \in F_\delta\} \subseteq U_\delta$. For each t let $\tilde{V}_t = V_t^* \cup \{(X_\alpha \times X_\alpha)^* : \alpha \neq \delta, \alpha \in \Delta\}$ and let $\tilde{U}_\delta = U_\delta^* \cup \{(X_\alpha \times X_\alpha)^* : \alpha \neq \delta, \alpha \in \Delta\}$. With $p(\delta) = t$ and $p(\alpha)$ arbitrary in Γ_α for $\alpha \neq \delta$, \tilde{V}_t is in $\prod\{\mathcal{V}_{p(\alpha)} : \alpha \in \Delta\}$ and so in \mathcal{W} . Since the finite intersection $\cap\{\tilde{V}_t : t \in F_\delta\}$ is contained in \tilde{U}_δ , \tilde{U}_δ is in \mathcal{W} . Clearly $\cap\{\tilde{U}_\alpha : \alpha \in \Delta\}$ is contained in $\cup\{U_\alpha^* : \alpha \in \Delta\}$ and so the latter, since Δ is finite, is in \mathcal{W} . Thus, when Δ is finite, equality holds as asserted.

As [3] suggests, a pair (Y, f) , where Y is compact and T_2 , f is one-to-one into Y , and the image of f is dense in Y , can be considered a T_2 compactification associated with the set X , which is the domain of f . The topology (or uniformity) for X need not be explicitly mentioned because they are determined by Y and f . To say that the class of (Y, f) is the

supremum of the classes of a collection $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ implicitly assumes that each (Y_α, f_α) is also associated with X . This language is used in the last two results.

Lemma R24.17 For $1 \leq i \leq n$, let (Y_i, f_i) be a T_2 compactification associated with a set X_i . Let $g : \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n Y_i$ by $g(x, i) = (f_i(x), i)$. Then $(\prod_{i=1}^n Y_i, g)$ is a T_2 compactification associated with $\prod_{i=1}^n X_i$.

Proof: This is simply a restatement of R24.15ii in compactification-focused language.

Corollary R24.18 For $1 \leq i \leq n$, let (Y_i, f_i) be a T_2 compactification associated with a set X_i . Assume for each i there is a non-empty family of T_2 compactifications $\{(Y_\alpha^i, f_\alpha^i) : \alpha \in \Delta_i\}$ such that the class of (Y_i, f_i) is the supremum of the classes of $\{(Y_\alpha^i, f_\alpha^i) : \alpha \in \Delta_i\}$. Then the class of $(\prod_{i=1}^n Y_i, g)$, where $g : \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n Y_i$ by $g(x, i) = (f_i(x), i)$ is the supremum of the classes $\{(\prod_{i=1}^n Y_{p(i)}^i, g_p) : p \in \prod_{i=1}^n \Delta_i\}$, where $g_p(x, i) = (f_{p(i)}^i(x), i)$ as in R24.17.

Proof: For each i and each $\alpha \in \Delta_i$, let \mathcal{U}_α^i be the separated, totally bounded uniformity associated with $[(Y_\alpha^i, f_\alpha^i)]$. By R13.1.7 $\mathcal{U}_i = \vee \{\mathcal{U}_\alpha^i : \alpha \in \Delta_i\}$ is the uniformity associated with $[(Y_i, f_i)]$. By R24.15iii $\prod_{i=1}^n \mathcal{U}_i$ is the uniformity associated with $(\prod_{i=1}^n Y_i, g)$. For each $p \in \prod_{i=1}^n \Delta_i$, again by R24.15iii $\prod_{i=1}^n \mathcal{U}_{p(i)}^i$ is the uniformity associated with $(\prod_{i=1}^n Y_{p(i)}^i, g_p)$. The conclusion now follows from R13.1.7 and R24.16.

Using equivalence of compactifications allows a more concise statement of the last conclusion:

$$\left(\prod_{i=1}^n Y_i, g\right) \equiv \bigvee \left\{ \left(\prod_{i=1}^n Y_{p(i)}^i, g_p\right) : p \in \prod_{i=1}^n \Delta_i \right\}$$

Note that in the special case when all Δ_i except one are singletons, the last result could be described by saying the finite disjoint union is distributive over arbitrary suprema.

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<http://www.susanjkleinart.com/compactification/>

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