Uniformities and Normal Bases

Uniformity Generated by a Normal Basis

Definition R14.1.1 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and $\mathcal{Z}$ a normal basis for (X,τ). $\mathcal{B}(\mathcal{Z}) = \{\cup_{i\in\Delta}(X - Z_i) \times (X - Z_i) : \Delta \text{ is finite, } Z_i \text{ is in } \mathcal{Z}, \text{ and } \cap_{i\in\Delta} Z_i = \emptyset\}$.

Definition R14.1.2 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and $\mathcal{Z}$ a normal basis for (X,τ). $\mathcal{U}(\mathcal{B}(\mathcal{Z})) = \{U \subseteq X \times X : B \subseteq U \text{ for some } B \in \mathcal{B}(\mathcal{Z})\}$.

From [3] recall the map Ψ_0, which associates a totally bounded uniformity with the compactification class it generates. The next proposition shows that $\mathcal{U}(\mathcal{B}(\mathcal{Z}))$ is a uniformity and also that $\Psi_0(\mathcal{U}(\mathcal{B}(\mathcal{Z}))) = [(\omega(\mathcal{Z}), \iota_{\mathcal{Z}}]$. In other words, given a normal basis $\mathcal{Z}$, $\mathcal{U}(\mathcal{B}(\mathcal{Z}))$ is the uniformity generating an equivalent compactification.

Proposition R14.1.3 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and $\mathcal{Z}$ a normal basis for (X,τ). Let $\mathcal{U} \in \mathcal{TB}((X,\tau))$ such that $\Psi_0(\mathcal{U}) = [(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})]$. Then $\mathcal{U}(\mathcal{B}(\mathcal{Z})) = \mathcal{U}$.

Proof: By R1.6a, the hypotheses imply that $\iota_{\mathcal{Z}}$ is a uniform embedding from $(X,\mathcal{U})$ into $\omega(\mathcal{Z})$. Let $U \in \mathcal{U}$. There is an entourage V in the unique uniformity for $\omega(\mathcal{Z})$ such that $(\iota_{\mathcal{Z}} \times \iota_{\mathcal{Z}})[U] = (\iota_{\mathcal{Z}}[X] \times \iota_{\mathcal{Z}}[X]) \cap V$. Since $\{Z^\omega : Z \in \mathcal{Z}\}$ is a closed base for the compact $\omega(\mathcal{Z})$ and V is a neighborhood of the diagonal, there exist $Z_1, \ldots, Z_n$ in $\mathcal{Z}$ such that $\{\omega(\mathcal{Z}) - Z_i^\omega : i = 1, \ldots, n\}$ is a cover of $\omega(\mathcal{Z})$ and $\cup_{i=1}^n (\omega(\mathcal{Z}) - Z_i^\omega) \times (\omega(\mathcal{Z}) - Z_i^\omega) \subseteq V$. It follows that $\cap_{i=1}^n Z_i^\omega = \emptyset$ and so $\cap_{i=1}^n Z_i = \emptyset$. It is easy to check that $\cup_{i=1}^n (X - Z_i) \times (X - Z_i)$, an element of $\mathcal{B}(\mathcal{Z})$, is contained in U, i.e., $U \in \mathcal{U}(\mathcal{B}(\mathcal{Z}))$.

Conversely, let $B \in \mathcal{B}(\mathcal{Z})$, say $B = \cup_{i\in\Delta}(X - Z_i) \times (X - Z_i)$, where Δ is finite, Z_i is in $\mathcal{Z}$, and $\cap_{i\in\Delta} Z_i = \emptyset$. Then it follows that $\cap_{i\in\Delta} Z_i^\omega = \emptyset$ so that $\{\omega(\mathcal{Z}) - Z_i^\omega : i = 1, \ldots, n\}$ is a cover of $\omega(\mathcal{Z})$. Thus $W = \cup_{i\in\Delta}(\omega(\mathcal{Z}) - Z_i^\omega) \times (\omega(\mathcal{Z}) - Z_i^\omega)$ is in the unique uniformity for $\omega(\mathcal{Z})$. It is easy to check that $(\iota_{\mathcal{Z}} \times \iota_{\mathcal{Z}})^{-1}[W] \subseteq B$ and so $\mathcal{U}(\mathcal{B}(\mathcal{Z}))$ is contained in $\mathcal{U}$.

As an application we return to the example with $X_0 = [0,1]$, τ_0 discrete, and $\mathcal{U}_0 = \mathcal{U}_m \vee \mathcal{V}$ where $\mathcal{U}_m$ is the smallest element of $\mathcal{TB}((X_0,\tau_0))$, as in [3] the set of totally bounded uniformities generating τ_0, and $\mathcal{V}$ is the usual uniformity generated by the absolute value metric on X_0. Throughout this section X_0, τ_0 and $\mathcal{U}_0$ will always represent the structures just described.

In [5] it is shown that every zero-dimensional compactification of a discrete space can be generated by a normal basis and that the compactification associated with $\mathcal{U}_0$ is not zero-dimensional. The question of interest, which is still unanswered: Is the compactification associated with $\mathcal{U}_0$ generated by a normal basis ? A negative answer would settle the Frink question. (See subsection R9.3.) A positive answer might suggest a more general approach to it.

The next definition and lemma establish a notation and a convenient fact.

Definition R14.1.4 Let S be a co-finite subset of a set X. U_S is the set $S \times S \cup \{(x,x) : x \in X - S\}$.

Lemma R14.1.5 Let $(X,\mathcal{U})$ be a uniform space and $\mathcal{B}$ a base for $\mathcal{U}$. Let $\mathcal{U}_m$ be the smallest totally bounded uniformity for X with the discrete topology. Then $\{U_S \cap B : S$ is co-finite in X and $B \in \mathcal{B}\}$ is a base for $\mathcal{U}_m \vee \mathcal{U}$.

Proof: Given $W \in \mathcal{U}_m$, there exists S, a co-finite subset of X, such that $S \times S \subseteq W$. It follows that U_S, which is in $\mathcal{U}_m$, is a subset of W. Thus sets of the form U_S are basic in

$\mathcal{U}_m$. Since $\mathcal{B}$ is a base for $\mathcal{U}$, the conclusion follows from the definition of the supremum of two uniformities.

Proposition R14.1.6 Assume that $\mathcal{Z}$ is a normal basis for (X_0, τ_0) and that $\Psi_0(\mathcal{U}_0) = [(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})]$. Then

i) Given $\epsilon > 0$ and S co-finite in X_0, there is $B \in \mathcal{B}(\mathcal{Z})$ such that $B \subseteq V_\epsilon \cap U_S$.

ii) Given $B \in \mathcal{B}(\mathcal{Z})$, there is $\gamma > 0$ and T co-finite in X_0 such that $V_\gamma \cap U_T \subseteq B$.

Proof: By R14.1.3 $\mathcal{U}_0 = \mathcal{U}(\mathcal{B}(\mathcal{Z}))$. Since entourages of the form $V_\epsilon \cap U_S$ are basic in $\mathcal{U}_0$ by R14.1.5, both assertions follow immediately.

Generating a Normal Basis from a Uniformity

It is natural to ask whether, given a totally bounded separated uniformity $\mathcal{U}$ for X, there is a normal basis $\mathcal{Z}$ for X such that $\Psi_0(\mathcal{U}) = [(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})]$. Of course, this is just the Frink question in different langauge.

The approach examined here is motivated by P3.14, the fact that the zero-sets of a $T_{3\frac{1}{2}}$ space form a normal basis which generates the Stone-Čech compactification. Since every continuous map into $[0, 1]$ extends to the Stone-Čech compactification, by R7.1.1 every continuous map is uniformly continuous, if on the domain one uses the totally bounded separated uniformity corresponding to the Stone-Čech compactification. In this case, the set of zero-sets is identical to the set of uniform zero-sets as defined below.

Given a separated uniformity $\mathcal{U}$ on set X, we first ask: is the collection of uniform zero-sets of $(X,\mathcal{U})$ always a normal basis for X? That question is only partially answered because of a negative answer to a second question: if $\mathcal{U}$ is separated and totally bounded and the uniform-zero sets of $(X,\mathcal{U})$ do form a normal basis, is the compactification generated by the normal basis necessarily equivalent to the compactification generated by $\mathcal{U}$?

Definition R14.2.1 Let $\mathcal{U}$ be a uniformity for X, and let $[0, 1]$ have the uniformity from the absolute value metric. A set $A \subseteq X$ is a uniform zero-set of $(X,\mathcal{U})$ provided there exists $f : (X,\mathcal{U}) \to [0, 1]$ uniformly continuous such that $A = f^{-1}[\{0\}]$.

Definition R14.2.2 Let $\mathcal{U}$ be a uniformity for X. $\mathcal{Z}(\mathcal{U}) = \{A \subseteq X : A$ is a uniform zero-set of $(X,\mathcal{U})\}$.

The next lemma records that $\mathcal{Z}(\mathcal{U})$ must always satisfy the first three requirements for a normal basis.

Lemma R14.2.3 Let $\mathcal{U}$ be a uniformity for X. Then

i) $\mathcal{Z}(\mathcal{U})$ is a closed base for $(X, \tau(\mathcal{U})$.

ii) $\mathcal{Z}(\mathcal{U})$ is closed under finite unions and intersections.

iii) If F is $\tau(\mathcal{U})$-closed and $x \notin F$, then there exists $A \in \mathcal{Z}(\mathcal{U})$ such that $x \in A$ and $A \cap F = \emptyset$.

Proof: Recall the following facts, which assume the usual absolute value uniformity on $\mathbb{R}$, the real numbers: Addition is a uniformly continuous operation on $\mathbb{R}$, multplication is uniformly continuous on compact subsets of $\mathbb{R}$, and, if f is a uniformly continuous map into $\mathbb{R}$, then $f \wedge 1$ is also uniformly continuous. The elements of $\mathcal{Z}(\mathcal{U})$ are clearly closed in $(X, \tau(\mathcal{U}))$. Let F be closed in $(X, \tau(\mathcal{U}))$ with $x \in X - F$. There is $U \in \mathcal{U}$ such that $U[x] \subseteq X - F$. By R7.2.13, there is a pseudo-metric d on X such that $U \in \mathcal{U}_d$, the uniformity generated by d, and $\mathcal{U}_d \subseteq \mathcal{U}$. Using R7.2.14, we have $f(t) = d(t, F) \wedge 1$ is uniformly continuous from $(X,\mathcal{U})$ into $[0, 1]$. Thus $f^{-1}[\{0\}]$ is in $\mathcal{Z}(\mathcal{U})$. It is easy to check that $F \subseteq f^{-1}[\{0\}]$ and $x \notin f^{-1}[\{0\}]$ so that i) holds. For the same F, x, U and d, $g(t) = d(t,x) \wedge 1$ is uniformly continuous from $(X,\mathcal{U})$ into $[0,1]$, $x \in g^{-1}[\{0\}]$, and $F \cap g^{-1}[\{0\}] = \emptyset$. Thus iii) holds. Finally let $A = f^{-1}[\{0\}]$ and $B = g^{-1}[\{0\}]$, where f and g are uniformly continuous from $(X,\mathcal{U})$ into $[0,1]$. By the facts on operations $f \cdot g$ and $h = \frac{1}{2}(f+g)$ are also uniformly continuous. Since $A \cup B = (f \cdot g)^{-1}[\{0\}]$ and $A \cap B = h^{-1}[\{0\}]$, property ii) holds.

Proposition R14.2.4 Let d be a metric for X and let $\mathcal{U}_d$ be the uniformity generated by d. Then

i) $\mathcal{Z}(\mathcal{U}_d)$ is a normal basis for $(X, \tau(\mathcal{U}_d)$.

ii) $\omega(\mathcal{Z}(\mathcal{U}_d))$ is equivalent to the Stone-Čech compactification of $(X, \tau(\mathcal{U}_d)$.

Proof: Let F be closed in $(X, \tau(\mathcal{U}_d)$. By R7.2.14 $f(x) = d(x,F) \wedge 1$ is uniformly continuous from $(X,\mathcal{U}_d)$ into $[0,1]$. Since $F = f^{-1}[\{0\}]$, every closed set is in $\mathcal{Z}(\mathcal{U}_d)$. The fourth requirement for a normal basis follows easily from topological normality, and so i) holds. This also shows that $\mathcal{Z}(\mathcal{U}_d)$ is the set of ordinary zero-sets, a normal basis which generates the Stone-Čech compactification, and so ii) holds.

Example R14.2.5 Let $X = (0,1]$ and let d be the absolute value metric. $\mathcal{U}_d$ is totally bounded and $\Psi_0(\mathcal{U}_d)$ is the equivalence class of the one-point compactification, which is not equivalent to the Stone-Čech compactification, i.e., the compactification associated with $\mathcal{U}_d$ is not equivalent to $\omega(\mathcal{Z}(\mathcal{U}_d))$.

The idea of uniform zero-sets is very similar to (and perhaps equivalent to) an idea briefly discussed by Chandler [1; p. 97]. He uses an example not unlike R14.2.5 to show that it does not answer the Frink question.

More about the Example

This subsection is essentially an addendum to [5], with emphsis on the example X_0 etc. as described above.

The following results use two normal bases, $\mathcal{Z}_1$ and $\mathcal{Z}_2$, simultaneously, and so some notational complications are needed. For x in the underlying space, $\hat{x}^1$ and $\hat{x}^2$ will denote the point filters of x in $\omega(\mathcal{Z}_1)$ and $\omega(\mathcal{Z}_2)$ respectively. For $A \in \mathcal{Z}_1$, the associated basic closed set is $A^{\omega_1} = \{\mathcal{F} \in \omega(\mathcal{Z}_1) : A \in \mathcal{F}\}$. For $B \in \mathcal{Z}_2$ the analogous set will be denoted B^{ω_2}.

Lemma R14.3.1 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Suppose $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are normal bases for (X, τ) with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Assume $A \in \mathcal{Z}_1$ and $X - A \in \mathcal{Z}_2$. If $\mathcal{G} \cap \mathcal{Z}_1 \in \omega(\mathcal{Z}_1)$ for every $\mathcal{G} \in \omega(\mathcal{Z}_2)$, then A^{ω_1} is clopen in $\omega(\mathcal{Z}_1)$.

Proof: First, note that, being members of the closed basis for $\omega(\mathcal{Z}_2)$, A^{ω_2} and $(X-A)^{\omega_2}$ are both closed in $\omega(\mathcal{Z}_2)$. In addition, $A^{\omega_2} \cup (X-A)^{\omega_2} = \omega(\mathcal{Z}_2)$ by P3.3 and $A^{\omega_2} \cap (X-A)^{\omega_2} = \emptyset$ by the definition of a $\mathcal{Z}_2$-filter. Thus $A^{\omega_2} = \omega(\mathcal{Z}_2) - (X-A)^{\omega_2}$ and so A^{ω_2} is clopen in $\omega(\mathcal{Z}_2)$. Next, as in the proof of R9.1.1, $\phi(\mathcal{G}) = \mathcal{G} \cap \mathcal{Z}$ defines the continuous surjection from $\omega(\mathcal{Z}_2)$ to $\omega(\mathcal{Z}_1)$ such that $\phi(\hat{x}^2) = \hat{x}^1$ for every x. Clearly $\omega(\mathcal{Z}_1) = \phi[A^{\omega_2}] \cup \phi[(X-A)^{\omega_2}]$. Now suppose $\phi[A^{\omega_2}] \cap \phi[(X-A)^{\omega_2}]$ is non-empty, and let $\mathcal{H}$ be in the intersection, i.e., $\mathcal{H} = \phi(\mathcal{F}) = \phi(\mathcal{G})$ where $A \in \mathcal{F}$ and $X - A \in \mathcal{G}$. Then $\mathcal{H} = \mathcal{F} \cap \mathcal{Z}_1 = \mathcal{G} \cap \mathcal{Z}_1$ so that A is also in $\mathcal{G}$, which contradicts the definition of a $\mathcal{Z}_2$-filter. Thus the intersection is empty and so $\phi[A^{\omega_2}]$ is clopen in $\omega(\mathcal{Z}_1)$. To finish, we verify $\phi[A^{\omega_2}] = A^{\omega_1}$. If $\mathcal{H} = \mathcal{G} \cap \mathcal{Z}_1$ for some $\mathcal{G}$ in $\omega(\mathcal{Z}_2)$ with $A \in \mathcal{G}$, then clearly $A \in \mathcal{H}$, i.e., $\mathcal{H} \in A^{\omega_1}$. Conversely, given $\mathcal{H} \in A^{\omega_1}$, since ϕ is onto, there is $\mathcal{G}$ in $\omega(\mathcal{Z}_2)$ with $\mathcal{H} = \mathcal{G} \cap \mathcal{Z}_1$.

Clearly $A \in \mathcal{G}$, i.e., $\mathcal{G} \in A^{\omega_2}$.

Lemma R14.3.2 Let (X,τ) be a $T_{3\frac{1}{2}}$ space. Suppose $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are normal bases for (X,τ) with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Assume $\omega(\mathcal{Z}_1) \leq \omega(\mathcal{Z}_2)$. Let $\phi : \omega(\mathcal{Z}_2) \to \omega(\mathcal{Z}_1)$ be the unique continuous surjection with $\phi(\hat{x}^2) = \hat{x}^1$ for all $x \in X$. Then, for $\mathcal{G} \in \omega(\mathcal{Z}_2)$, $\phi(\mathcal{G})$ is the unique $\mathcal{Z}_1$-ultrafilter containing $\mathcal{G} \cap \mathcal{Z}_1$, i.e., $\phi(\mathcal{G}) = \{B \in \mathcal{Z}_1 : B \cap A \neq \emptyset \text{ for all } A \in \mathcal{G} \cap \mathcal{Z}_1\}$.

Proof: By R9.1.3 and R9.1.4 the conclusion follows if $\mathcal{G} \cap \mathcal{Z}_1 \subseteq \phi(\mathcal{G})$ for all $\mathcal{G} \in \omega(\mathcal{Z}_2)$. Let $\mathcal{G} \in \omega(\mathcal{Z}_2)$ and suppose $Z \in \mathcal{G} \cap \mathcal{Z}_1$ but $Z \notin \phi(\mathcal{G})$. By P3.3 there is $A \in \phi(\mathcal{G})$ such that $A \cap Z = \emptyset$. By P3.1iv there are $C, D \in \mathcal{Z}_1$ with $C \cup D = X$, $A \subseteq X - C$, and $Z \subseteq X - D$. Note that $C \notin \phi(\mathcal{G})$ since $A \cap C = \emptyset$ and $D \notin \mathcal{G}$ since $D \cap Z = \emptyset$. Next use density to find a net $\{x_\alpha\}$ in X such that $\hat{x}^2_\alpha \to \mathcal{G}$. By continuity $\hat{x}^1_\alpha \to \phi(\mathcal{G})$. Since $\phi(\mathcal{G})$ is in the $\omega(\mathcal{Z}_1)$-open set $\omega(\mathcal{Z}_1) - C^{\omega_1}$, there is an α_1 such that $\alpha \geq \alpha_1$ implies $\hat{x}^1_\alpha \notin C^{\omega_1}$, i.e., $x_\alpha \notin C$. Since $\mathcal{G}$ is in the $\omega(\mathcal{Z}_2)$-open set $\omega(\mathcal{Z}_2) - D^{\omega_2}$, there is an α_2 such that $\alpha \geq \alpha_2$ implies $\hat{x}^2_\alpha \notin D^{\omega_2}$, i.e., $x_\alpha \notin D$. For any α greater than or equal both α_1 and α_2, $x_\alpha \notin C \cup D = X$, a contradiction.

The next proposition uses the notation $E(A)$, which was introduced in [6] and denotes $A \times A \cup (X - A) \times (X - A)$ for A contained in X.

Proposition R14.3.3 Let $A \subseteq X_0$. Then $E(A) \in \mathcal{U}_0$ if and only if A is either finite or co-finite.

Proof: If A is finite or co-finite, then $E(A) \in \mathcal{U}_m$ and the desired conclusion follows from the definition of $\mathcal{U}_0$. Now assume $E(A) \in \mathcal{U}_0$. Since $\{V_\epsilon : \epsilon > 0\}$ is a base for $\mathcal{V}$, by R14.1.5 there is $\epsilon > 0$ and S, a co-finite subset of X_0, with $U_S \cap V\epsilon \subseteq E(A)$ and so $(U_S \cap V_\epsilon)^n \subseteq E(A)^n = E(A)$ for any integer n. By R9.3.6 $(U_S \cap V_\epsilon)^n = U_S \cap V_{n\epsilon}$. Pick n so that $n\epsilon > 1$. Then $V_{n\epsilon} = X_0 \times X_0$ and so $U_S \subseteq E(A)$. It follows easily that $S \subseteq A$ or $S \subseteq X_0 - A$.

The last proposition provides an example not included in [6]. In notation defined in [6], it implies that $\mathcal{U}(\mathcal{R}(\mathcal{U}_0)) = \mathcal{U}_m$. Thus $\mathcal{U}(\mathcal{R}(\mathcal{U}_0)) \in \mathcal{TB}((X_0, \tau_0))$ but $\mathcal{U}(\mathcal{R}(\mathcal{U}_0))$ is a proper subset of $\mathcal{U}_0$.

Proposition R14.3.4 Assume $\mathcal{Z}_1$ is a normal basis for (X_0, τ_0) such that $\Psi_0(\mathcal{U}_0) = [(\omega(\mathcal{Z}_1), \iota_{\mathcal{Z}_1})]$ and let $A \in \mathcal{Z}_1$. Suppose $\mathcal{Z}_2$ is also a normal basis for (X_0, τ_0) with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ and $X - A \in \mathcal{Z}_2$. If $\mathcal{G} \cap \mathcal{Z}_1 \in \omega(\mathcal{Z}_1)$ for every $\mathcal{G} \in \omega(\mathcal{Z}_2)$, then $E(A) \in \mathcal{U}_0$.

Proof: By R14.3.1, A^{ω_1} is a clopen subset of $\omega(\mathcal{Z}_1)$. As a result, $V = A^{\omega_1} \times A^{\omega_1} \cup ((\omega(\mathcal{Z}_1) - A^{\omega_1}) \times (\omega(\mathcal{Z}_1) - A^{\omega_1}))$ is an open neighborhood of the diagonal for $\omega(\mathcal{Z}_1)$ and so V is in the unique uniformity for $\omega(\mathcal{Z}_1)$. By R1.6a the embedding $\iota_{\mathcal{Z}_1}$ is uniformly continuous from $(X_0, \mathcal{U}_0)$ so that $(\iota_{\mathcal{Z}_1} \times \iota_{\mathcal{Z}_1})^{-1}[V] \in \mathcal{U}_0$. It is routine to check that $(\iota_{\mathcal{Z}_1} \times \iota_{\mathcal{Z}_1})^{-1}[V] = E(A)$.

Proposition R14.3.5: Suppose $\mathcal{Z}$ is a normal basis for (X_0, τ_0) such that $\Psi_0(\mathcal{U}_0) = [(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})]$. Let $A \subseteq X_0$. If both A and $X_0 - A$ are in $\mathcal{Z}$, then A must be either finite or co-finite.

Proof: Apply R14.3.4 with $\mathcal{Z}_1 = \mathcal{Z}_2 = \mathcal{Z}$ to conclude that $E(A) \in \mathcal{U}_0$. The conclusion is immediate from R14.3.3.

Proposition R14.3.6: Suppose $\mathcal{Z}_1$ is a normal basis for (X_0, τ_0) such that $\Psi_0(\mathcal{U}_0) = [(\omega(\mathcal{Z}_1), \iota_{\mathcal{Z}_1})]$. Assume $\mathcal{Z}_2$ is also a normal basis for (X_0, τ_0) with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Then either $\mathcal{G} \cap \mathcal{Z}_1 \notin \omega(\mathcal{Z}_1)$ for some $\mathcal{G} \in \omega(\mathcal{Z}_2)$ or $Z \in \mathcal{Z}_1$ and $X_0 - Z \in \mathcal{Z}_2$ imply Z is either finite or co-finite.

Proof: Assume $\mathcal{G} \cap \mathcal{Z}_1 \in \omega(\mathcal{Z}_1)$ for all $\mathcal{G} \in \omega(\mathcal{Z}_2)$ and let $Z \in \mathcal{Z}_1$ with $X_0 - Z \in \mathcal{Z}_2$. By R14.3.4 $E(A) \in \mathcal{U}_0$ and by R14.3.3 Z is either finite or co-finite.

Discussion Assume $\mathcal{Z}_1$ is a normal basis for (X_0, τ_0) with $\Psi_0(\mathcal{U}_0) = [(\omega(\mathcal{Z}_1), \iota_{\mathcal{Z}_1})]$. Since $\mathcal{U}_m$ is strictly smaller than $\mathcal{U}_0$, by R1.5 the compactification associated with $\mathcal{U}_m$ (the one point compactification) is not equivalent to $\omega(\mathcal{Z}_1)$. Because the normal basis of finite and co-finite sets generates the one point compactification of (X_0, τ_0), $\mathcal{Z}_1$ must contain a set which is neither finite nor co-finite.

Now let $\mathcal{Z}_2$ be the set of all zero-sets of X_0. Since (X_0, τ_0) is discrete, $\mathcal{Z}_2$ is the power set of X_0 and $\mathcal{Z}_2$-ultrafilters are simply ordinary ultrafilters. By R14.3.6 there would have to be an ultrafilter $\mathcal{G}$ such that $\mathcal{G} \cap \mathcal{Z}_1$ is a prime $\mathcal{Z}_1$-filter but not a $\mathcal{Z}_1$-ultrafilter. (This would be an example not provided in the first subsection of [5].) Moreover, the argument of R14.3.4 shows that, if A in $\mathcal{Z}_1$ is neither finite nor co-finite, then A^{ω_1} is not clopen in $\omega(\mathcal{Z}_1)$. Finally, by P3.14 $\omega(\mathcal{Z}_2)$ is the Stone-Čech compactification, and so $\omega(\mathcal{Z}_1) \leq \omega(\mathcal{Z}_2)$. Since the conclusion of R14.3.1 would not hold for $A \in \mathcal{Z}_1$ neither finite nor co-finite, the hypothesis '$\mathcal{G} \cap \mathcal{Z}_1 \in \omega(\mathcal{Z}_1)$ for every $\mathcal{G} \in \omega(\mathcal{Z}_2)$' in R14.3.1 could not be weakened to '$\omega(\mathcal{Z}_1) \leq \omega(\mathcal{Z}_2)$.' R14.3.2 would apply, but the argument of R14.3.1 would fail because, given $A \in \mathcal{Z}_1$ and ϕ as in R14.3.1, the conclusion of R14.3.2 is not enough to show that $\phi[A^{\omega_2}] \cap \phi[(X - A)^{\omega_2}] = \emptyset$.

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References

1. Chandler, R.E., Hausdorff Compactifications, Marcel Dekker, Inc., 1976.
2. This website, P3: Normal Bases
3. This website, R1: Existence of Suprema via Uniform Space Theory
4. This website, R7: Uniform Continuity and Extension of Maps
5. This website, R9: Directed Sets of Normal Bases
6. This website, R11: The Magill-Glasenapp Theorem

Added Comments 2007

Chandler and Faulkner [7] state that the Frink question was resolved by Ul'yanov [8], who proved that "the assertion that every Hausdorff bicompact extension of an arbitrary separable completely regular space is an extension of Wallman type is equivalent to the continuum hypothesis."

Additional References

An asterisk indicates a reference not seen by me.

7. Chandler, R. E. and Faulkner, G.D., Hausdorff Compactifications: A Retrospective, in: A Handbook of the History of General Topology, vol. 2, C.E. Aull and R. Lowen, editors, Kulwer Academic Publishers, 1998.

8* Ul'yanov, V.M., Solution of the Fundamental Problem of Bicompact Extensions of Wallman Type, Soviet Math. Dokl. 233(1977), 567-571.

Added 2023

Some results in [5] can be strengthened and extended by using R14.1.3 and the generalized notions of related compactifications and suprema in [9].

Proposition R14.Add.1 Let τ_1, τ_2 be $T_{3\frac{1}{2}}$ topologies for a set X. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases for (X, τ_1) and (X, τ_2) respectively. Assume $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$.
Then $(\omega(\mathcal{Z}_1), \iota_{\mathcal{Z}_1}) \leq (\omega(\mathcal{Z}_2), \iota_{\mathcal{Z}_2})$.

Proof: Let $\mathcal{U}_1, \mathcal{U}_2$ be the separated totally bounded uniformities with $\mathcal{U}_i$ corresponding to the compactification class of $(\omega(\mathcal{Z}_i), \iota_{\mathcal{Z}_i})$. By R14.1.3 $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ implies $\mathcal{U}_1 \subseteq \mathcal{U}_2$. The conclusion now follows from R13.1.2.

The following improves R9.1.1iii

Corollary R14.Add.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Then $(\omega(\mathcal{Z}_1), \iota_{\mathcal{Z}_1}) \leq (\omega(\mathcal{Z}_2), \iota_{\mathcal{Z}_2})$.

Proof: This specializes R14.Add.1 with $\tau = \tau_1 = \tau_2$.

Lemma R14.Add.3 Let $\{\tau_\delta : \delta \in \Delta\}$ be a non-empty collection of $T_{3\frac{1}{2}}$ topologies for a set X. For each δ assume that $\mathcal{Z}_\delta$ is a normal basis for (X, τ_δ) and that $\{\mathcal{Z}_\delta : \delta \in \Delta\}$ is a directed set relative to containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\delta : \delta \in \Delta\}$. Then

i) $\{\tau_\delta : \delta \in \Delta\}$ is a directed set relative to containment.

ii) $\mathcal{Z}$ is a normal basis for $(X, \vee\{\tau_\delta : \delta \in \Delta\})$.

Proof: Because each normal basis is a closed base for the associated topology, $\mathcal{Z}_\alpha \subseteq \mathcal{Z}_\beta$ implies $\tau_\alpha \subseteq \tau_\beta$. As a result the assumed directed set property is inherited by the set of topologies, i.e., i) holds. For ii) note first that the directed set assumption implies that any finite subset of $\mathcal{Z}$ is also contained in $\mathcal{Z}_\delta$ for some δ. Thus P3.1ii and P3.1iv hold for $\mathcal{Z}$ because they hold for each $\mathcal{Z}_\delta$. Each $Z \in \mathcal{Z}$ is closed relative to $\vee\{\tau_\delta : \delta \in \Delta\}$ because each Z must be in some $\mathcal{Z}_\alpha$ and so closed relative to τ_α, which is contained in the supremum. Now let E be closed in the supremum with $x \notin E$. Then there exist $O_1, \ldots, O_n$ in $\cup\{\tau_\delta : \delta \in \Delta\}$ such that $x \in \cap_{i=1}^n O_i \subseteq X - E$. By the directed set property there is γ such that $O_i \in \tau_\gamma$ for $i = 1, \ldots, n$. Since $\mathcal{Z}_\gamma$ is a closed base for τ_γ, there is $Z \in \mathcal{Z}_\gamma \subseteq \mathcal{Z}$ such that $x \notin Z$ and $E \subseteq X - (\cap_{i=1}^n O_i) \subseteq Z$. Thus $\mathcal{Z}$ is a closed base for the supremum, i.e., P3.1i holds. Note for the same $x, E, O_1, \ldots, O_n$ and γ, since $\mathcal{Z}_\gamma$ is a normal basis for τ_γ, there is $W \in \mathcal{Z}_\gamma \subseteq \mathcal{Z}$ such $x \in W$ and $W \cap (X - \cap_{i=1}^n O_i) = \emptyset$ The last implies $W \cap E = \emptyset$ so that P3.1.iii holds for $\mathcal{Z}$. Conclusion ii) follows from P3.1, the definition of a normal basis.

Comment: Part i) of the previous lemma also follows from R14.Add.1 and R13.1.5i.

Lemma R14.Add.4 Let X be a set and let $\{\mathcal{U}_\delta : \delta \in \Delta\}$ be a non-empty collection of uniformities for X which is a directed set relative to containment.
Then $\vee\{\mathcal{U}_\delta : \delta \in \Delta\} = \cup\{\mathcal{U}_\delta : \delta \in \Delta\}$.

Proof: For every $\alpha \in \Delta$, $\mathcal{U}_\alpha \subseteq \vee\{\mathcal{U}_\delta : \delta \in \Delta\}$ and so the union is contained in the supremum. Now let $U \in \vee\{\mathcal{U}_\delta : \delta \in \Delta\}$. Since finite intersections from the union are a base for the supremum, there are $U_1, \ldots, U_j$ in the union with $\cap_{i=1}^j U_i \subseteq U$. By the directed set assumption there is $\alpha \in \Delta$ with $U_i \in \mathcal{U}_\alpha$ for $i = 1, \ldots, j$. Because uniformities are closed under finite intersections and supersets, U is in $\mathcal{U}_\alpha$ and so in the union.

The next example, which is a bit of a digression, shows that a parallel version of R14.Add.4 for a directed set of topologies does not hold in general. In the example the directed set is a chain in which each topology is neither T_2 nor completely regular.

Example R14.Add.5 For each $n \in \mathbb{N}$ let τ_n be the collection containing $\mathbb{N}$ and all subsets of $\{1, 2, \ldots, n\}$. Clearly each τ_n is a topology for $\mathbb{N}$ and $\tau_n \subseteq \tau_{n+1}$. Here $\vee\{\tau_n : n \in \mathbb{N}\}$ is discrete because every singleton is open. But $\cup\{\tau_n : n \in \mathbb{N}\}$ is not a topology because it is not closed under arbitrary unions: $\mathbb{N}$ is the only infinite subset in the union.

Proposition R14.Add.6 Let $\{\tau_\delta : \delta \in \Delta\}$ be a non-empty collection of $T_{3\frac{1}{2}}$ topologies for a set X. For each δ assume that $\mathcal{Z}_\delta$ is a normal basis for (X, τ_δ) and that $\{\mathcal{Z}_\delta : \delta \in \Delta\}$ is a directed set relative to containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\delta : \delta \in \Delta\}$. Then $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ acts as a supremum for $\{(\mathcal{Z}_\delta, \iota_{\mathcal{Z}_\delta}) : \delta \in \Delta\}$.

Proof: R14.Add.3ii shows that $\mathcal{Z}$ is a normal basis for $(X, \vee\{\tau_\delta : \delta \in \Delta\})$. For each δ in Δ let $\mathcal{U}_\delta$ be the separated totally bounded uniformity associated with the compactification class of $(\omega(\mathcal{Z}_\delta), \iota_{\mathcal{Z}_\delta})$ and let $\mathcal{U}$ be the separated totally bounded uniformity associated with the compactification class of $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. R14.Add.1 shows that $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is an upper bound for $\{(\omega(\mathcal{Z}_\delta), \iota_{\mathcal{Z}_\delta}) : \delta \in \Delta\}$ and so by R13.1.2 $\mathcal{U}_\delta \subseteq \mathcal{U}$ for every $\delta \in \Delta$. By R14.1.3, $\mathcal{Z}_\gamma \subseteq \mathcal{Z}_\delta$ implies $\mathcal{U}_\gamma \subseteq \mathcal{U}_\delta$ and so $\{\mathcal{U}_\delta : \delta \in \Delta\}$ is a directed set relative to containment. Let $U \in \mathcal{U}$. By R14.1.3 and the directed set hypothesis, $U \in \cup\{\mathcal{U}_\delta : \delta \in \Delta\}$, which is $\vee\{\mathcal{U}_\delta : \delta \in \Delta\}$ by R14.Add.4. Thus $\mathcal{U} = \vee\{\mathcal{U}_\delta : \delta \in \Delta\}$. The conclusion now follows from R13.1.7.

In R9.2.4 the underlying topological space is assumed to be discrete and the normal bases are all obtained from some n-compatible equivalence relation on X. The following corollary shows that those limiting assumptions can be removed.

Corollary R14.Add.7 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\{\mathcal{Z}_\delta : \delta \in \Delta\}$ be a non-empty collection of normal bases for (X, τ) which is a directed set relative to containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\delta : \delta \in \Delta\}$. Then $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ acts as a supremum for $\{(\mathcal{Z}_\delta, \iota_{\mathcal{Z}_\delta}) : \delta \in \Delta\}$.

Proof: This is a special case of the previous result, with $\tau_\delta = \tau$ for every $\delta \in \Delta$.

Additional Reference

9. This website, R13: Mixed Suprema