The Magill-Glasenapp Theorem

Among the results in [3] is the following:

**Theorem** [Magill and Glasenapp] Let \((X, \tau)\) be \(T_3\) and zero-dimensional. The supremum of any non-empty collection of zero-dimensional \(T_2\) compactifications of \(X\) is also zero-dimensional.

The keys to the proof in [3] are two observations. First, the clopen subsets of a 0-dimensional compactification induce a Boolean ring of clopen subsets of the base space. Secondly, the induced Boolean ring can be used to construct a compactification equivalent to the original.

This section will give a uniformity-based proof the theorem stated above. The key, the construction of \(\mathcal{M}(U)\), is largely derived from [3]. Notation, definitions, and facts from [4] will be used freely.

**Definition R11.1** [2] Let \(X\) be a set and let \(E\) be an equivalence relation on \(X\). \(U_E = \{U \subseteq X \times X : E \subseteq U\}\).

Since \(E \circ E = E\), it is clear that \(U_E\) is a uniformity for \(X\). The uniform space \((X, U_E)\) will be called the uniform space generated by \(E\).

**Definition R11.2** [2] Let \((X, U)\) be a uniform space. \((X, U)\) is an \(e\)-uniform space provided \((X, U)\) is the supremum of some non-empty family of uniform spaces generated by equivalence relations.

Given an \(e\)-uniform space \((X, U)\), \(U\) will be called an \(e\)-uniformity. The following proposition is immediate from the definition.

**Proposition R11.3** (Levine [2]) The supremum of a family of \(e\)-uniformities for \(X\) is also an \(e\)-uniformity for \(X\).

**Proposition R11.4** (Levine [2]) Let \((X, U)\) be an \(e\)-uniform space. Then \((X, \tau(U))\) is 0-dimensional.

Proof: For an equivalence relation \(E\) on \(X\), the equivalence classes are clopen in \((X, \tau(U_E))\). Such clopen sets form a basis for the topology of a supremum.

**Definition R11.5** Let \(X\) be a set and let \(A \subseteq X\). \(E(A) = A \times A \cup (X - A) \times (X - A)\).

**Lemma R11.6** Let \(X\) be a set and let \(A_1\) and \(A_2\) be subsets of \(X\). Then \(E(A_1) \cap E(A_2)\) is a subset of both \(E(A_1 \cup A_2)\) and \(E(A_1 \cap A_2)\).

Proof: Routine.

**Definition R11.7** Let \((X, U)\) be a uniform space. \(\mathcal{R}(U) = \{A \subseteq X : E(A) \in U\}\).

**Lemma R11.8** Let \((X, U)\) be a uniform space. Then

i) If \(A \in \mathcal{R}(U)\), then \(A\) is clopen relative to \((X, \tau(U))\).

ii) \(\emptyset\) and \(X\) are in \(\mathcal{R}(U)\).

iii) If \(A \in \mathcal{R}(U)\), then \(X - A \in \mathcal{R}(U)\).

iv) \(\mathcal{R}(U)\) is closed under finite unions and finite intersections.

Proof: i) holds since \(E(A)[x] = A\) if \(x \in A\) and \(E(A)[x] = X - A\) if \(x \in X - A\). ii) is clear and iii) holds since \(E(X - A) = E(A)\). iv) follows from R11.6.

Next we recall the definition of algebraic operations on \(\mathcal{P}(X)\), the power set of \(X\).

**Definition R11.9** Let \(X\) be a set. For \(A, B \in \mathcal{P}(X)\), \(A + B = (A \cup B) - (A \cap B)\) and \(A \cdot B = AB = A \cap B\).
As is well-known, $(\mathcal{P}(X), +, \cdot)$ is a commutative ring with multiplicative identity $X$. The additive identity is $\emptyset$, the negative of $A$ is $A$ since $A + A = \emptyset$, and the ring is Boolean since $A^2 = A$. Because $-A = A$, algebraic subtraction need not be used, although the algebraic difference $X - A = X + (-A) = X + A$ is the complement of $A$ in $X$ so that $X - A$ unambiguously denotes the complement of $A$.

**Proposition R11.10** Let $(X, \mathcal{U})$ be a uniform space. Then $(\mathcal{R}(\mathcal{U}), +, \cdot)$ is a commutative ring with unity.

**Proof:** By R11.8.i and iv, $\mathcal{R}(\mathcal{U})$ is closed under $\cdot$ and contains the multiplicative identity. Since $A + B = (A \cup B) \cap (X - (A \cap B))$, R11.8.iii and iv show $\mathcal{R}(\mathcal{U})$ is also closed under $+$. 

**Definition R11.11** Let $(X, \mathcal{U})$ be a uniform space. $\mathcal{M}(\mathcal{U})$ denotes the set of maximal ideals in $\mathcal{R}(\mathcal{U})$. For $x \in X$, $M_x = \{A \in \mathcal{R}(\mathcal{U}) : x \notin A\}$.

**Lemma R11.12** Let $(X, \mathcal{U})$ be a uniform space. Then

i) If $x \in X$, then $M_x \in \mathcal{M}(\mathcal{U})$.

ii) If $M \in \mathcal{M}(\mathcal{U})$ and $A \in \mathcal{R}(\mathcal{U})$, then either $A \in M$ or $X - A \in M$.

**Proof:** Clearly $\emptyset \in M_x$ and, since $A + B \subseteq A \cup B$, $M_x$ is closed under addition. For $A \in M_x$ and $C \in \mathcal{R}(\mathcal{U})$, since $CA \subseteq A$, $CA \in M_x$. Thus $M_x$ is an ideal. Now let $I$ be an ideal of $\mathcal{R}(\mathcal{U})$ with $M_x \subseteq I$. Suppose $B \in I$ but $B \notin M_x$. Then $x \in B$ and so $X - B \in M_x$. Thus $B + (X - B) = X \in I$ and so $I = \mathcal{R}(\mathcal{U})$. For the second assertion, let $M \in \mathcal{M}(\mathcal{U})$ and $A \in \mathcal{R}(\mathcal{U})$ be given. Note that $A(X - A) = 0$ in $\mathcal{R}(\mathcal{U})$. In the field $\mathcal{R}(\mathcal{U})/M$, $(A + M)((X - A) + M) = 0 + M$, which implies $A \in M$ or $X - A \in M$.

**Definition R11.13** Let $(X, \mathcal{U})$ be a uniform space, and let $A \in \mathcal{R}(\mathcal{U})$. $H_A = \{M \in \mathcal{M}(\mathcal{U}) : A \in M\}$.

**Lemma R11.14** Let $(X, \mathcal{U})$ be a uniform space. Then

i) For $A \in \mathcal{R}(\mathcal{U})$, $H_{X - A} = \mathcal{M}(\mathcal{U}) - H_A$.

ii) For $A, B \in \mathcal{R}(\mathcal{U})$, $H_{AB} = H_A \cup H_B$.

iii) There is a topology $\sigma$ on $\mathcal{M}(\mathcal{U})$ with $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$ as a base for the closed sets.

iv) For each $A \in \mathcal{R}(\mathcal{U})$, $H_A$ is clopen in $\sigma$.

v) $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$ is also an open basis for $\sigma$.

**Proof:** For i), note that $H_A \cup H_{X - A} = \mathcal{M}(\mathcal{U})$ by R11.12.ii, and $H_A \cap H_{X - A} = \emptyset$ since $A + (X - A)$ equals the multiplicative identity, which can’t be in a maximal ideal. The first assertion is immediate from these equations. For ii), let $A, B \in \mathcal{R}(\mathcal{U})$. If $M \in H_A \cup H_B$, then $A$ or $B$ is in $M$ so that $AB$ is in the ideal $M$, i.e. $M \in H_{AB}$. Now let $M \in H_{AB}$ and suppose $M \notin H_A$. By R11.12.ii the complement of $A$, i.e. $1 + A$, must be in $M$, as is $B(1 + A) + AB = B + AB + AB = B$. Thus $M \in H_B$. For iii), first note that $H_X = \emptyset$ and $H_{\emptyset} = \mathcal{M}(\mathcal{U})$. By ii) $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$ is closed under finite unions. These facts imply that there is a unique topology on $\mathcal{M}(\mathcal{U})$ with $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$ as a base for its closed sets. Part iv) is now immediate from i). By part i) and R11.8.iii $\{H_A : A \in \mathcal{R}(\mathcal{U})\} = \{\mathcal{M}(\mathcal{U}) - H_A : A \in \mathcal{R}(\mathcal{U})\}$. Since the complements of a base for the closed sets must form an open basis, v) holds.

For the remainder of this section, $\mathcal{M}(\mathcal{U})$ will always be assumed to have the zero-dimensional topology with clopen basis $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$. The next proposition, a version of a result which is credited in [3] to Gillman and Jerison [1; p 111,7M], implies by P2.4
that $\mathcal{M}(\mathcal{U})$ has a unique uniformity, the neighborhoods of the diagonal.

**Proposition R11.15** Let $(X, \mathcal{U})$ be a uniform space. Then $\mathcal{M}(\mathcal{U})$ is compact and $T_2$.

**Proof:** Let $\{H_A : A \in S\}$ be a non-empty family with the finite intersection property. For compactness it is sufficient to show that $\cap \{H_A : A \in S\} \neq \emptyset$. Let $I = \{\sum_{i=1}^{n} B_i A_i : B_i \in \mathcal{R}(\mathcal{U}) \text{ and } A_i \in S\}$. It is easily checked that $I$ is an ideal in $\mathcal{R}(\mathcal{U})$ and $S \subseteq I$. If $X$ is in $I$, say $X = \sum_{i=1}^{n} B_i A_i$, apply the finite intersection property to select $N \in \cap \{H_{A_i} : i = 1, \ldots, n\}$. Then each $A_i$ is in $N$ and so is $\sum_{i=1}^{n} B_i A_i = X$, a contradiction since every maximal ideal is proper. Thus $X \notin I$ and so there is a maximal ideal $M$ with $I \subseteq M$. Since $S \subseteq I$, $A \in M$ for all $A$ in $S$, i.e., $M \in \cap \{H_A : A \in S\}$. For $T_2$, let $M, N \in \mathcal{M}(\mathcal{U})$ with $M \neq N$. Pick $A \in M$ with $A \notin N$. Then $M$ is in the clopen $H_A$ and $N$ is not.

**Definition R11.16** Let $(X, \mathcal{U})$ be a uniform space.

$\mathcal{U}(\mathcal{R}(\mathcal{U})) = \forall \{\mathcal{U}_{E(A)} : A \in \mathcal{R}(\mathcal{U})\}$.

**Lemma R11.17** Let $(X, \mathcal{U})$ be a uniform space. Then

i) $\mathcal{U}(\mathcal{R}(\mathcal{U}))$ is a totally bounded e-uniformity for $X$.

ii) $\mathcal{U}(\mathcal{R}(\mathcal{U})) \subseteq \mathcal{U}$.

iii) $\mathcal{R}(\mathcal{U}(\mathcal{R}(\mathcal{U}))) = \mathcal{R}(\mathcal{U})$.

**Proof:** Since each $\mathcal{U}_{E(A)}$ is totally bounded and the supremum of totally bounded uniformities is also totally bounded by P2.13, i) is clear from the definition of an e-uniformity. Part ii) holds since each $\mathcal{U}_{E(A)}$ is contained in $\mathcal{U}$ by definition of $\mathcal{R}(\mathcal{U})$. Part iii) is easy to check.

**Definition R11.18** Let $(X, \mathcal{U})$ be a uniform space. $\mu : X \to \mathcal{M}(\mathcal{U})$ is defined by $\mu(x) = M_x$.

**Lemma R11.19** Let $(X, \mathcal{U})$ be a uniform space. Then

i) If $A \in \mathcal{R}(\mathcal{U})$, then $\mu^{-1}[H_A] = X - A$.

ii) $\mu : (X, \mathcal{U}(\mathcal{R}(\mathcal{U}))) \to \mathcal{M}(\mathcal{U})$ is uniformly continuous.

iii) $\mu[X]$ is dense in $\mathcal{M}(\mathcal{U})$.

**Proof:** Let $A \in \mathcal{R}(\mathcal{U})$. Following the definitions, one easily sees that $\mu^{-1}[H_A] = \{x \in X : A \in M_x\} = X - A$, and so i) holds. For ii), let $V$ be a neighborhood of the diagonal of $\mathcal{M}(\mathcal{U})$. Using compactness and the basis for $\mathcal{M}(\mathcal{U})$, there exist $A_1, \ldots, A_j$ in $\mathcal{R}(\mathcal{U})$ such that $\cup_{i=1}^{j} H_{A_i} \times H_{A_i}$ is an open neighborhood of the diagonal and is contained in $V$. By i) $\cup_{i=1}^{j} (X - A_i) \times (X - A_i) \subseteq (\mu \times \mu)^{-1}[V]$. Since $H_{A_1}, \ldots, H_{A_j}$ cover $\mathcal{M}(\mathcal{U})$, $\cup_{i=1}^{j} (X - A_i) = X$. By definition $\cap_{i=1}^{j} E(A_i) \in \mathcal{U}(\mathcal{R}(\mathcal{U}))$ and it is easy to check that $\cap_{i=1}^{j} E(A_i) \subseteq \cap_{i=1}^{j} (X - A_i) \times (X - A_i)$ so that $(\mu \times \mu)^{-1}[V]$ is in $\mathcal{U}(\mathcal{R}(\mathcal{U}))$, as required for uniform continuity. For iii) consider a basic open set $H_A \neq \emptyset$ with $A \in \mathcal{R}(\mathcal{U})$. Since $H_X = \emptyset$, $A \neq X$ so that $\mu[X] \cap H_A = \{M_x : x \notin A\}$ is non-empty.

**Lemma R11.20** Let $(X, \mathcal{U})$ be a uniform space. Then

i) $\mu$ is one-to-one if and only if $\mathcal{U}(\mathcal{R}(\mathcal{U}))$ is separated.

ii) If $\mathcal{U}(\mathcal{R}(\mathcal{U}))$ is separated, then $\mu : (X, \mathcal{U}(\mathcal{R}(\mathcal{U}))) \to \mathcal{M}(\mathcal{U})$ is a uniform embedding onto $\mu[X]$.

**Proof:** First assume $\mu$ is one-to-one and let $a, b$ be in $X$ with $a \neq b$. Since $M_a \neq M_b$, there is $A \in \mathcal{R}(\mathcal{U})$ with $a \in A$ and $b \notin A$. Then $(a, b) \notin E(A)$ so that $(a, b) \notin \cap\{U : U \in \mathcal{U}(\mathcal{R}(\mathcal{U}))\}$, i.e., $\mathcal{U}(\mathcal{R}(\mathcal{U}))$ is separated. For the converse, let $a, b$ be in $X$
with $a \neq b$. Since $(a, b) \notin \cap\{U : U \in \mathcal{U}(\mathcal{R}(\mathcal{U}))\}$, there exist $A_1, \ldots, A_j$ in $\mathcal{R}(\mathcal{U})$ such that $(a, b) \notin \cap_{i=1}^j E(A_i)$. Pick $i$ with $(a, b) \notin E(A_i)$. Then $A_i$ is in one of $M_a, M_b$ and not the other, i.e., $M_a \neq M_b$. Thus $\mu$ is one-to-one. For ii), because of the first part and R11.19ii, it is only necessary to show that $\mu \times \mu[U]$ is an entourage in the subspace uniformity on $\mu[X]$ for any $U \in \mathcal{U}(\mathcal{R}(\mathcal{U}))$. Since $\mu \times \mu[E(A)] = (\mu[X] \times \mu[X]) \cap ((H_{X-A} \times H_{X-A}) \cup (H_A \times H_A))$ and $\mu \times \mu$ is one-to-one, this follows easily from the definition of $\mathcal{U}(\mathcal{R}(\mathcal{U}))$.

**Definition R11.21** Let $(X, \mathcal{U})$ be a uniform space. $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$ if and only if $\mathcal{U} = \mathcal{U}(\mathcal{R}(\mathcal{U}))$.

**Lemma R11.22** Let $(X, \mathcal{U})$ be a uniform space. Then $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$ if and only if $\mathcal{U}$ is a totally bounded e-uniformity.

**Proof:** The necessity is immediate from R11.17i. Now assume $\mathcal{U} = \mathcal{U}(\mathcal{R}(\mathcal{U}))$ is totally bounded, where $\mathcal{E}$ is a non-empty family of equivalence relations on $X$. For $E \in \mathcal{E}$, $\mathcal{U}_E$ is also totally bounded and so $E$ has finitely many distinct equivalence classes $C_1, \ldots, C_j$. Since $E \subseteq E(C_i)$, each $C_i$ is in $\mathcal{R}(\mathcal{U})$. It is easy to check that $E = \cap_{i=1}^j E(C_i)$ and so $E \in \mathcal{U}(\mathcal{R}(\mathcal{U}))$. The conclusion follows.

In the next few results some notation from [5] will be used. Given a $T_{3\frac{1}{2}}$ space $(X, \tau)$, $\mathcal{T}(X)$ will denote the set of totally bounded uniformities that generate $\tau$ and, for $\mathcal{U} \in \mathcal{T}(X)$, $\Psi_0(\mathcal{U})$ is the class of $T_2$ compactifications determined by $\mathcal{U}$.

**Proposition R11.23** Let $(X, \mathcal{U})$ be a uniform space. If $\mathcal{U}$ is separated and $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$, then

i) $(\mathcal{M}(\mathcal{U}), \mu)$ is a $T_2$ compactification of $(X, \tau(\mathcal{U}))$.

ii) $\Psi_0(\mathcal{U}) = [(\mathcal{M}(\mathcal{U}), \mu)]$.

**Proof:** With the given assumptions, part i) is immediate from R11.15, R11.19iii, and R11.20ii. Let $\mathcal{V}$ in $\mathcal{T}(X)$ be such that $\Psi_0(\mathcal{V}) = [(\mathcal{M}(\mathcal{U}), \mu)]$. By R1.6a, $\mathcal{V}$ makes $\mu$ a uniform embedding, where $\mu[X]$ has the subspace uniformity from the unique uniformity of $\mathcal{M}(\mathcal{U})$. $\mathcal{U}$ also makes $\mu$ a uniform embedding so that $\mathcal{U} = \mathcal{V}$.

The final two results could be stated without assuming that $X$ is zero-dimensional, since that is implied by the existence of a zero-dimensional $T_2$ compactification of $X$. As usual, a slightly loose set-like expression is used to indicate a supremum of compactifications.

**Proposition R11.24** Let $(X, \tau)$ be a zero-dimensional $T_{3\frac{1}{2}}$ space, let $(Y, f)$ be a zero-dimensional $T_2$ compactification of $X$, and let $\mathcal{U} \in \mathcal{T}(X)$ be the uniformity with $\Psi_0(\mathcal{U}) = [(Y, f)]$. Then $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$ and $(\mathcal{M}(\mathcal{U}), \mu)$ is equivalent to $(Y, f)$.

**Proof:** First note, for any $C$ clopen in $Y$, $A = f^{-1}[C]$ is clopen in $X$. In addition, $(C \times C) \cup ((Y - C) \times (Y - C))$ is in the unique uniformity for $Y$ and $E(A) = (f \times f)^{-1}[(C \times C) \cup ((Y - C) \times (Y - C))]$. Since $f : (X, \mathcal{U}) \to Y$ is uniformly continuous, $E(A) \in \mathcal{U}$ and $A \in \mathcal{R}(\mathcal{U})$. Now let $U \in \mathcal{U}$. There is $V$, a neighborhood of the diagonal in $Y$, such that $(f \times f)[U] = (f[X] \times f[X]) \cap V$. By the compactness and zero-dimensionality of $Y$, there exist clopen sets $C_1, \ldots, C_n$ of $Y$ such that $Y = \cup_{i=1}^n C_i$ and $\cap_{i=1}^n C_i \times C_i \subseteq V$. Let $A_i = f^{-1}[C_i]$. It is easy to check that $\cap_{i=1}^n E(A_i) \subseteq U$, which implies $U \in \mathcal{U}(\mathcal{R}(\mathcal{U}))$. Thus $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$. By R11.23ii $\Psi_0(\mathcal{U}) = [(\mathcal{M}(\mathcal{U}), \mu)]$ so that $(\mathcal{M}(\mathcal{U}), \mu)$ is equivalent to $(Y, f)$.

**Theorem R11.25** [Magill-Glasenapp] Let $(X, \tau)$ be a zero-dimensional $T_{3\frac{1}{2}}$ space. Let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ be a non-empty family of zero-dimensional $T_2$ compactifications of
and let \((Y,f) = \vee\{(Y_\alpha,f_\alpha) : \alpha \in \Delta\}\). Then \((Y,f)\) is a zero-dimensional \(T_2\) compactification of \(X\).

Proof: For each \(\alpha\), let \(U_\alpha\) be the uniformity in \(TB(X)\) with \(\Psi_0(U_\alpha) = [(Y_\alpha,f_\alpha)]\). Let \(U = \vee\{U_\alpha : \alpha \in \Delta\}\). By R1.5 \(\Psi_0(U) = [(Y,f)]\). By R11.24 and R11.22 each \(U_\alpha\) is a totally bounded \(e\)-uniformity. By P2.13 and R11.3 \(U\) is also a totally bounded \(e\)-uniformity so that, by R11.22 again, \(R(U)\) generates \(U\). By R11.23 \([(Y,f)] = [(M(U),\mu)]\). Since \(M(U)\) is zero-dimensional, so is \(Y\).

Albert J. Klein 2005
http://www.susanjkleinart.com/compactification/

References


4. This website, P2: Uniform Spaces

5. This website, R1: Existence of the Supremum via Uniform Space Theory

Added Comments 2009

Herrlich and Strecker [6, p. 315] state that MacGill and Glasenapp rediscovered a result known earlier. Apparently it was originally due to Banaschewski [7] in 1955.

Additional References

An asterisk indicates a reference not seen by me.
