Uniform Spaces

This website will use the entourage or Bourbaki approach to uniform spaces as presented in Kelley [2]. The equivalent Tukey approach via uniform coverings, which some find more intuitive, can be found in Isbell [1].

**Definition P2.1** Let $X$ be a set. A uniformity for $X$ is $\mathcal{U}$, a non-empty set of relations on $X$, such that

i) $\forall U \in \mathcal{U}, \{(x, x) : x \in X\} \subseteq U$

ii) $U \in \mathcal{U}$ and $U \subseteq W \subseteq X \times X \Rightarrow W \in \mathcal{U}$

iii) $U \in \mathcal{U}$ and $W \in \mathcal{U} \Rightarrow U \cap W \in \mathcal{U}$

iv) $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$

v) $U \in \mathcal{U} \Rightarrow \exists V \in \mathcal{U}$ with $V \circ V \subseteq U$

The set $\{(x, x) : x \in X\}$ may be referred to as the diagonal of $X$ or $\Delta_X$. The pair $(X, \mathcal{U})$ is called a uniform space. $(X, \mathcal{U})$ and the uniformity $\mathcal{U}$ are called separated provided $\bigcap\{U : U \in \mathcal{U}\} = \Delta_X$.

A motivating example for definition P2.1 arises when $X$ has a pseudo-metric $\rho$. For $\epsilon > 0$ let $V_\epsilon = \{(x, y) : \rho(x, y) < \epsilon\}$. Let $\mathcal{U}_\rho = \{U \subseteq X \times X : \exists \epsilon > 0 \text{ with } V_\epsilon \subseteq U\}$. Then $\mathcal{U}_\rho$ is a uniformity, which is separated if and only if $\rho$ is a metric.

**Definition P2.2** Let $(X, \mathcal{U})$ be a uniform space. $\tau(\mathcal{U}) = \{G \subseteq X : x \in G \Rightarrow \exists U \in \mathcal{U} \text{ with } U[x] \subseteq G\}$.

$\tau(\mathcal{U})$ is a topology for $X$. A topological space $(X, \tau)$ is called uniformizable provided there exists a uniformity $\mathcal{U}$ for $X$ with $\tau(\mathcal{U}) = \tau$. When $X$ has a pseudo-metric $\rho$, $\tau(\mathcal{U}_\rho)$ is the topology generated by the pseudo-metric. In general, there need not be a unique uniformity generating a given uniformizable topology. A uniformity $\mathcal{U}$ is separated if and only if $\tau(\mathcal{U})$ is $T_2$.

**Theorem P2.3** A topological space is uniformizable if and only if it is completely regular. It is uniformizable via a separated uniformity if and only if it is $T_{3\frac{1}{2}}$.

**Proposition P2.4** Let $(X, \tau)$ be compact and $T_2$. Then there exists a unique uniformity $\mathcal{U}$ such that $\tau(\mathcal{U}) = \tau$. Moreover, $\mathcal{U}$ is the set of all neighborhoods of the diagonal in $X \times X$.

**Definition P2.5** Let $(X, \mathcal{U})$ be a uniform space, and let $S : D \to X$ be a net. $S$ is Cauchy if and only if $\forall U \in \mathcal{U}, \exists d_0 \in D$ so that $d, e \geq d_0 \Rightarrow (S(d), S(e)) \in U$. $(X, \mathcal{U})$ is complete if and only if every Cauchy net converges in $(X, \tau(\mathcal{U}))$.

**Definition P2.6** Let $(X, \mathcal{U})$ be a uniform space. $(X, \mathcal{U})$ is totally bounded if and only if $\forall U \in \mathcal{U}, \exists x_1, ..., x_n \in X$ such that $X \subseteq \bigcup_{i=1}^{n} U[x_i]$.

When $X$ has a pseudo-metric $\rho$, these definitions of complete and totally bounded for $(X, \mathcal{U}_\rho)$ are equivalent to the usual definitions based on the pseudo-metric. For $A \subseteq X$, $A$ is totally bounded provided $A$ with the subspace uniformity from $X$ is totally bounded. Every subset of a totally bounded set is totally bounded. If $A$ is totally bounded, then $\overline{A}$ is also totally bounded, where $\overline{A}$ denotes the closure of $A$ in $(X, \tau(\mathcal{U}))$.

**Theorem P2.7** Let $(X, \mathcal{U})$ be a uniform space. Then $(X, \tau(\mathcal{U}))$ is compact if and only if $(X, \mathcal{U})$ is complete and totally bounded.

**Definition P2.8** Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces, and let $f : X \to Y$. $f$ is uniformly continuous if and only if $\forall V \in \mathcal{V}, (f \times f)^{-1}[V] \in \mathcal{U}$. 

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When $X$ and $Y$ have pseudo-metrics $\rho$ and $\sigma$, uniform continuity relative to $(X, U_\rho)$ and $(Y, V_\sigma)$ is equivalent to the usual definition of uniform continuity based on the pseudo-metrics. A map $f$ as in P2.8 is called a unimorphism provided $f$ is a bijection, $f$ is uniformly continuous, and $f^{-1}$ is uniformly continuous. $f$ is called a uniform embedding provided $f$ is a unimorphism from $X$ to $f[X]$, where $f[X]$ has the subspace uniformity.

**Proposition P2.9** Let $(X, U)$ and $(Y, V)$ be uniform spaces, and let $f : X \to Y$. If $f$ is uniformly continuous, then $f$ is continuous relative to $(X, \tau(U))$ and $(Y, \tau(V))$. If $f$ is continuous and $(X, \tau(U))$ is compact, then $f$ is uniformly continuous.

**Definition P2.10** Let $(X, U)$ be a uniform space. A completion of $(X, U)$ is a pair $((Y, V), f)$, where $(Y, V)$ is a complete uniform space, $f : X \to Y$ is a uniform embedding, and $f[X]$ is dense in $Y$.

**Theorem P2.11** Every uniform space has a completion. Every separated uniform space has a separated completion, which is unique up to unimorphism.

**Proposition P2.12** Let $\{U_\alpha : \alpha \in \Delta\}$ be a non-empty family of uniformities on $X$. Then there is a uniformity $U$ on $X$ such that $U_\alpha \subseteq U \ \forall \alpha \in \Delta$ and, if $V$ is a uniformity for $X$ with $U_\alpha \subseteq V \ \forall \alpha \in \Delta$, then $U \subseteq V$.

The uniformity $U$ is the supremum of $\{U_\alpha : \alpha \in \Delta\}$. The notations used will be $U = \bigvee \{U_\alpha : \alpha \in \Delta\}$ or $(X, U) = \bigvee \{(X, U_\alpha) : \alpha \in \Delta\}$.

**Proposition P2.13** Let $\{U_\alpha : \alpha \in \Delta\}$ be a non-empty family of uniformities on $X$. If $(X, U_\alpha)$ is totally bounded $\forall \alpha \in \Delta$, then $\bigvee \{(X, U_\alpha) : \alpha \in \Delta\}$ is also totally bounded.

**Proposition P2.14** Let $\{U_\alpha : \alpha \in \Delta\}$ be a non-empty family of uniformities on $X$. Then $\tau(\bigvee \{U_\alpha : \alpha \in \Delta\}) = \bigvee \{\tau(U_\alpha) : \alpha \in \Delta\}$.

Albert J. Klein 2003
http://www.susanjkleinart.com/compactification/

**References**